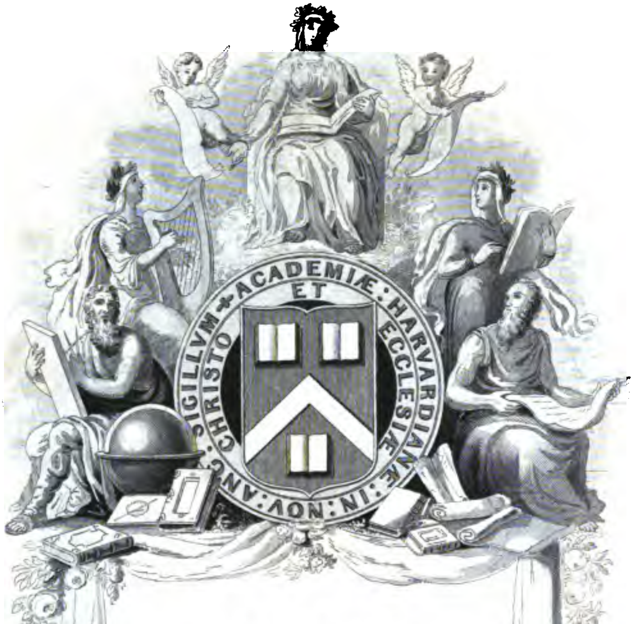
SIGILLVM ACADEMIÆ HARVARDIANÆ IN NOV: ANG:
CHRISTO ET ECCLESIÆ

# MATHEMATICAL QUESTIONS,

WITH THEIR

# SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

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## Mathematical Papers, &c.



## Solved Questions.

# CORRIGENDA.

## VOL. XV.

p. vi, Question 3024, *for* $n^4$ *read* $n^3$.

p. 52, lines 15 and 16, *for* $x_2-$, $y_2-$ *read* $x_2=$, $y_2=$.

p. 92, line 12, *for* $g$ *read* $q$.

## VOL. XVI.

p. 78, equation 4, *for* $(1-2e\cos\psi+e^2)$ *read* $(1-2e^3-e\cos\psi)$.

p. 79, line 19, *for* point of contact of *read* foot of the perpendicular from the origin on.

## VOL. XVII.

p. 24, line 6 from bottom, *for* $\mathrm{CF}-3a$ *read* $\mathrm{CF}=3a$.

p. 49, line 8, *for* Quest. 2652 *read* Quest. 3652.

p. 71, equations (A) and (B), *for* $z$ *read* $y$.

p. 72, line 4, *for* $=-3(\phi_1+\&c.)=0$ *read* $=m\pi-3n\pi=n\pi$.

p. 72, after line 5 add the following note:—It will be seen, by considering particular points on the ellipse, that $n$ in parts 1 and 2 must be an even number.

p. 86, line 2 of Solution, *for* on *read* in.

p. 86, line 4 of the verses on the right, *for* tried *read* tired.

p. 92, line 6 from bottom, *for* $-$ (between the fractions) *read* $=$.

# MATHEMATICS

FROM

## THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

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**3541.** (Proposed by the EDITOR.)—A piece of slender uniform wire, of indefinite length, every point of which attracts with a force varying according to the inverse square of the distance, is bent into the form of the catacaustic of the parabola for rays perpendicular to the axis, (or what is the same curve, the first negative focal pedal of the parabola); and $A_1$, $A_2$, $A_3$ are the several attractions on a particle at the focus, of the whole curve, of the loop of it, and of the part which exerts the greatest attraction towards the vertex; prove (1) that $A_1 : A_2 : A_3 = 8 : 6\sqrt{3} : 19$, and (2) that an arc of the curve extending from the vertex to an angular distance from the focus of 23° 44′ 2″ on each side of the axis would exert the same attraction as the whole curve on a particle at the focus.

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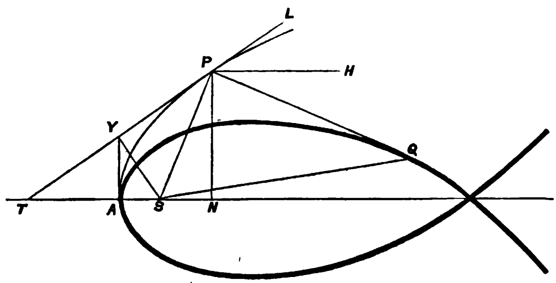
*Solution by* G. S. CARR.




1. Let A be the vertex of the parabola, S the focus, $AS = a$, PT a tangent, PH a diameter, SY perpendicular to PT, $\angle TSY = PSY = \alpha$.

Then, if Q be the point of ultimate intersection of reflected rays at P when the incident rays are perpendicular to the axis of the parabola, the locus of Q will be the catacaustic. Now, by the law of reflexion, $\angle$QPL = NPT; and by the property of the parabola, $\angle$HPL = SPT; therefore SPQ = NPH = a right angle, which shows the identity of the catacaustic and the first negative focal pedal of the parabola.

Let SQ $= r$, and $\angle$ASQ $= \theta$, then the equation to PQ is

$$r\cos(\theta - 2\alpha) = a\sec^2\alpha \qquad (1).$$

Differentiating for the parameter $\alpha$, we have

$$r\sin(\theta - 2\alpha) = a\sec^3\alpha\sin\alpha \qquad (2).$$

Eliminating $\alpha$, we have $r^2 = a^2\sec^6\alpha$, or $r = a\sec^3\alpha$;
thus (2) becomes $\sin(\theta - 2\alpha) = \sin\alpha$, whence $\theta = 3\alpha$;
and therefore the polar equation to the pedal or catacaustic is, by (1),

$$r = a\sec^3\tfrac{1}{3}\theta \qquad (3).$$

The equation (3) to the curve may be otherwise obtained by observing that, Q being the point on the pedal corresponding to P on the parabola, the triangles ASY, YSP, PSQ are similar; therefore $\theta =$ ASQ $= 3$ ASY $= 3\alpha$.

From (3) we have $\dfrac{dr}{d\theta} = r\tan\tfrac{1}{3}\theta$, $\therefore$ $\dfrac{ds}{d\theta} = \left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}} = r\sec\tfrac{1}{3}\theta$.

The resultant attraction of an element of the curve, when resolved horizontally towards the vertex, will be

$$\int\frac{\mu}{r^2}\cos\theta\,ds = \int\frac{\mu}{a}\cos^2\tfrac{1}{3}\theta\cos\theta\,d\theta = \frac{\mu}{2a}\int(1+\cos\tfrac{2}{3}\theta)\cos\theta\,d\theta$$
$$= \frac{\mu}{2a}\int(\cos\theta + \tfrac{1}{2}\cos\tfrac{5}{3}\theta + \tfrac{1}{2}\cos\tfrac{1}{3}\theta)\,d\theta$$
$$= \frac{\mu}{2a}(\sin\theta + \tfrac{3}{10}\sin\tfrac{5}{3}\theta + \tfrac{3}{2}\sin\tfrac{1}{3}\theta) \qquad (4).$$

Now $A_1 =$ (4), taken between limits 0 and $\frac{3}{2}\pi$ for the whole curve, $= \frac{8}{5}\left(\dfrac{\mu}{a}\right)$,

$A_2 =$ (4), taken between limits 0 and $\pi$ for the loop, $= \frac{3}{10}\sqrt{3}\left(\dfrac{\mu}{a}\right)$,

$A_3 =$ (4), taken between limits 0 and $\frac{1}{2}\pi$ for the part between the latus rectum and the vertex, $= \frac{19}{20}\left(\dfrac{\mu}{a}\right)$;

therefore $\quad A_1 : A_2 : A_3 = 8 : 6\sqrt{3} : 19.$

2. Here we must have, if $\theta$ be the angle required,

$$\int_0^\theta \cos^2\tfrac{1}{3}\theta\cos\theta = \tfrac{4}{5}.$$

This gives, if we write $x$ for $\sin\frac{1}{3}\theta$,

$$12x^5 - 25x^3 + 15x - 2 = (x-1)^2(12x^3 + 24x^2 + 11x - 2) = 0 \qquad (5).$$

The only real root of the cubic is

$$x = \tfrac{1}{6}\left\{(20+5\sqrt{11})^{\frac{1}{3}} + (20-5\sqrt{11})^{\frac{1}{3}} - 4\right\} = \cdot 1376397 = \sin 7^\circ 54' 40''\cdot 64;$$

therefore $\quad \theta = 23^\circ 44' 2''.$

**3230.** (Proposed by J. J. SYLVESTER, F.R.S.)—Given

$$\begin{aligned} x(x+y)(x+z)(x+t) &= i(x-y)(x-z)(x-t), \\ y(y+x)(y+z)(y+t) &= j(y-x)(y-z)(y-t), \\ z(z+x)(z+y)(z+t) &= k(z-x)(z-y)(z-t), \\ t(t+x)(t+y)(t+z) &= l(t-x)(t-y)(t-z); \end{aligned}$$

show that the system of values $x, y, z, t$, which satisfy the above equations, is *unique;* and determine $x, y, z, t$ in terms of $i, j, k, l$.

*Solution by the* PROPOSER.

The system of values required is

$$x = (i+j+k+l)\frac{i^3}{(i-j)(i-k)(i-l)}, \quad y = (i+j+k+l)\frac{j^3}{(j-i)(j-k)(j-l)},$$

$$z = (i+j+k+l)\frac{k^3}{(k-i)(k-j)(k-l)}, \quad t = (i+j+k+l)\frac{l^3}{(l-i)(l-k)(l-j)};$$

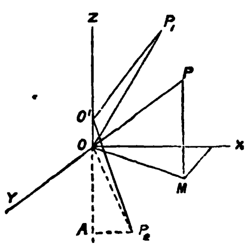
and it will be observed that $x+y+z+t = i+j+k+l$.

**3540.** (Proposed by the Rev. G. H. HOPKINS, M.A.)—A piece of wire every point of which attracts with a force varying as the inverse square of the distance, is twisted into a helix, $P_1$ and $P_2$ being the highest and lowest points; a particle $O'$, attached to the lowest point A of the axis by an extensible string, slides without friction along the axis (gravity not acting). If the modulus of elasticity be the attraction of the helix on the particle at O, when AO is the natural length of the string; prove that

$$OO' = AO\left(\frac{1}{O'P_2} - \frac{1}{O'P_1}\right) \div \left(\frac{1}{OP_2} - \frac{1}{OP_1}\right),$$

AO being less than one-half the length of the axis.

*Solution by the* PROPOSER.




The equations to the helix are $x^2+y^2 = a^2$, $z = s\tan\alpha$, $s$ being the length of the circular arc in the plane XY, measured from the point where the curve cuts the plane; and if $\sigma$ along the helix corresponds with $s$ in the plane, then $z = \sigma\sin\alpha$, $OP^2 = a^2+z^2 = a^2+\sigma^2\sin^2\alpha$.

The attraction of the element P upon O is $\frac{M}{OP^2}$, where M is some constant which is the same for every point on the curve; and the resolved part of this along OZ is

$$M\frac{\sigma\sin\alpha\, d\sigma}{(a^2+\sigma^2\sin^2\alpha)^{\frac{3}{2}}} = \frac{M}{\sin\alpha}\cdot\frac{dOP}{OP^2};$$

therefore the attractions along the axis of $z$ of the parts of the helix which lie above and below it, will be $\frac{M}{\sin\alpha}\left(\frac{1}{a}-\frac{1}{OP_1}\right)$, $\frac{M}{\sin\alpha}\left(\frac{1}{a}-\frac{1}{OP_2}\right)$, respectively. The entire force which acts upon O to draw it upwards, will be the difference between these, that is

$$\frac{M}{\sin\alpha}\left(\frac{1}{OP_2}-\frac{1}{OP_1}\right)=A_z.$$

When O takes another position, as O′, its change being due to the extension of the string, the force will be

$$\frac{M}{\sin\alpha}\left(\frac{1}{O'P_2}-\frac{1}{O'P_1}\right)=A'_z.$$

But $$AO'-AO=\frac{A'_z}{\lambda}.AO=\frac{A'_z}{A_z}.AO;$$

therefore $$OO'=AO\left(\frac{1}{O'P_2}-\frac{1}{O'P_1}\right)+\left(\frac{1}{OP_2}-\frac{1}{OP_1}\right).$$

---

**3539.** (Proposed by J. W. L. GLAISHER, B.A., F.R.A.S.)—Show that

$$\int_0^\infty \frac{dx}{e^{ue^x}+e^{-ue^x}}=-\mathrm{E}i(-u)+\mathrm{E}i(-3u)-\mathrm{E}i(-5u)+\ldots\ldots,$$

$$\int_0^\infty \frac{dx}{e^{ue^x}-e^{-ue^x}}=-\mathrm{E}i(-u)-\mathrm{E}i(-3u)-\mathrm{E}i(-5u)-\ldots\ldots,$$

E$i$ denoting the exponential-integral. As a particular case, putting $u=1$,

$$\int_0^\infty \frac{dx}{e^{e^x}+e^{-e^x}}=\cdot 2073794\ldots,\qquad \int_0^\infty \frac{dx}{e^{e}-e^{-ex}}=\cdot 2205447\ldots.$$

---

*Solution by the* PROPOSER.

$$\mathrm{E}i(-u)=\int_\infty^u \frac{e^{-x}}{x}\,dx=\int_\infty^1 \frac{e^{-ux}}{x}\,dx=\int_\infty^0 e^{-ue^z}\,dz\quad(\text{putting } x=e^z)$$
$$=-\int_0^\infty e^{-ue^z}\,dx,$$

therefore $$-\mathrm{E}i(-u)+\mathrm{E}i(-3u)-\mathrm{E}i(-5u)+\ldots\ldots$$

$$=\int_0^\infty\left(e^{-ue^x}-e^{-3ue^x}+e^{-5ue^x}-\ldots\right)dx=\int_0^\infty \frac{e^{-ue^x}\,dx}{1+e^{-2ue^x}}=\int_0^\infty \frac{dx}{e^{ue^x}+e^{-ue^x}}.$$

The second integral is proved in a similar manner.

The numerical values of the integrals for any value of $u$ are found at once from the *Tables of the Exponential-Integral* (*Phil. Trans.*, 1870, p. 367), as E$i(-x)$ decreases very fast as $x$ increases.

**3584.** (Proposed by J. B. SANDERS.)—If a body be projected down a plane inclined at 30° to the horizon, with a velocity equal to three-fourths of that due to the vertical height of the plane, prove that the time of descent down the plane is equal to that of falling through its height.

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*Solution by* J. H. SLADE; Q. W. KING; A. MARTIN; *and others.*

Let $2l$ be the length of plane, and therefore $l$ its vertical height, $v$ the velocity due to the height; and therefore, $\frac{3}{4}v$ the velocity of projection; also let $t$ and $t_1$ be the times of descent down the plane and its vertical height respectively. Then we have

$$\tfrac{3}{4}vt + \tfrac{1}{4}gt^2 = 2l = gt_1^2 = vt_1;$$

whence $4g^2t^2 + 12gvt = 32gl = 16v^2$, $(2gt+3v)^2 = 25v^2$, and $gt = v = gt_1$; therefore $t = t_1$, which proves the theorem.

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**3511.** (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Straight lines drawn from the angles of a triangle to the vertices of equilateral triangles described on the opposite sides, intersect, three and three, in two points; if another triangle be formed by connecting these two points and the centre of the circumscribing circle, show that the distance between the centres of gravity of the two triangles is equal to one-third of radius of the circumscribing circle.

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I. *Solution by* R. W. GENESE, B.A.

The first part is a particular case of the following theorem:—If along the perpendiculars to the sides of a triangle ABC, drawn through the middle points D, E, F, lengths DL, EM, FN be taken proportional to the sides BC, CA, AB respectively, then AL, BM, CN will cointersect. For conceive these lines to represent forces; AL can be resolved into two, represented by AD and a parallel through A to DL equal to it; and similarly for BM, CN. But the system AD, BE, CF is known to be in equilibrium; and the above parallels, being the perpendiculars of ABC, cointersect; being proportional to the sides of ABC, and making the same angles, *inter se*, as these sides, these forces are in equilibrium. Thus the system AL, BM, CN is in equilibrium; therefore AL, BM, CN cointersect.

In the particular case of the question, AL, BM, CN are easily seen to be equal, and must therefore make angles of 120° with each other.

Let P, Q be the two points in question. Then, of the two angles BPC, QPC, one will be seen to be 60°, and the other 120°, in all cases; and similarly for the other angles.

Thus PQ is a diameter of a rectangular hyperbola passing through ABC; and N, the middle point of PQ, must lie on the nine-point circle of ABC.

If X be the centre of gravity of OPQ, 3GX will be the resultant of forces represented by GO, GP, GQ.

But the resultant of GP and GQ is 2GN.

The centre of gravity G of ABC is a centre of similitude of the nine-

point and circumscribing circles. If NG meet the circle ABC in Y, YG = 2GN.

The resultant of YG and GO is YO in magnitude; therefore 3GX = radius of circumscribing circle.

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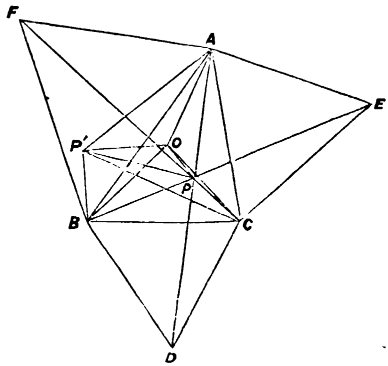
II. *Solution by* STEPHEN WATSON.

Let D, E, F be the *external* equilateral triangles, and let AD, BE intersect in P. Then the triangles BCE, DCA are equal in all respects, and $\angle PBC = \angle PDC$; hence a circle will pass through B, P, C, D and $\angle BPC = 120°$. From this it is plain that AD, BE, CF pass through the point P, such that $\angle BPC = CPA = APB = 120°$. Similarly when the equilateral triangles are described *internally* the three lines meet in a point P′ such that

$\angle AP'C = BP'C = 60°$.




Now we have

$$a^2 = BP^2 + CP^2 + BP.CP = BP^2 + CP^2 + \frac{4\Delta BPC}{\sqrt{3}}.$$

Similarly $\quad b^2 = CP^2 + AP^2 + \dfrac{4\Delta CPA}{\sqrt{3}}, \quad c^2 = AP^2 + BP^2 + \dfrac{4\Delta APC}{\sqrt{3}},$

therefore $\quad \frac{1}{2}(a^2 + b^2 + c^2) = AP^2 + BP^2 + CP^2 + \dfrac{2\Delta ABC}{\sqrt{3}}.$

Similarly $\quad \frac{1}{2}(a^2 + b^2 + c^2) = AP'^2 + BP'^2 + CP'^2 - \dfrac{2\Delta ABC}{\sqrt{3}},$

therefore $\quad AP^2 + BP^2 + CP^2 + AP'^2 + BP'^2 + CP'^2 = a^2 + b^2 + c^2 \quad .........(1).$

Let O be the circumscribing centre of the triangle ABC, and G, $g$ the centres of gravity of the triangles ABC, OPP′; then

$$\begin{aligned}
3GO^2 &= OA^2 + OB^2 + OC^2 - \tfrac{1}{3}(a^2 + b^2 + c^2) \\
3GP^2 &= AP^2 + BP^2 + CP^2 - \tfrac{1}{3}(a^2 + b^2 + c^2) \\
3GP'^2 &= AP'^2 + BP'^2 + CP'^2 - \tfrac{1}{3}(a^2 + b^2 + c^2) \\
3Gg^2 &= GO^2 + GP^2 + GP'^2 - \tfrac{1}{3}(OP^2 + OP'^2 + PP'^2) \quad .........(2).
\end{aligned}$$

By the aid of (1), the three first of these give

$$GO^2 + GP^2 + GP'^2 = \tfrac{1}{3}(AO^2 + BO^2 + CO^2) = R^2.$$

Also by the prize question in the *Lady's and Gentleman's Diary* for 1864, $OP^2 + OP'^2 + PP'^2 = 2R^2$; hence (2) becomes

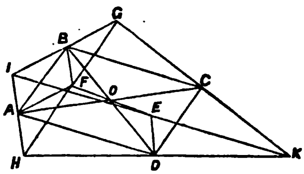
$$3Gg^2 = R^2 - \tfrac{2}{3}R^2 = \tfrac{1}{3}R^2, \quad \text{therefore } Gg = \tfrac{1}{3}R.$$

[Several other properties of the figure are investigated by Mr. WOOLHOUSE in his solution of the Diary Prize Question for 1865.]

**3607.** (Proposed by C. TAYLOR, M.A.)—Prove that the three straight lines joining the middle points of opposite sides and the middle points of the diagonals of a quadrilateral, bisect one another.

I. *Solution by* FRANCES E. PRUDDEN; T. MITCHESON, B.A.; J. H. SLADE; Q. W. KING; W. H. LAVERTY, M.A.; *and others.*

Let HIGK be a quadrilateral; A, B, C, D the middle points of the sides; F, E the middle points of the diagonals; then it is evident that ABCD is a parallelogram; therefore its diagonals AC and BD bisect one another in O; also DE is equal and parallel to BF; therefore FE and BD also bisect one another in O.




II. *Quaternion Solution by* Miss CLARKE.

Taking G as origin, let $\alpha$, $\beta$, $\gamma$ be the vectors of I, H, K; then the vectors of B, F, C, D, E, A will be, respectively, $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, $\frac{1}{2}\gamma$, $\frac{1}{2}(\beta+\gamma)$, $\frac{1}{2}(\gamma+\alpha)$, $\frac{1}{2}(\alpha+\beta)$; also the vector of the middle point O of AC is $\frac{1}{2}\left\{\frac{1}{2}(\alpha+\beta)+\frac{1}{2}\gamma\right\}$ or $\frac{1}{4}(\alpha+\beta+\gamma)$; and since this expression is symmetrical with respect to $\alpha$, $\beta$, $\gamma$, it follows that O is likewise the middle point of BD and EF.

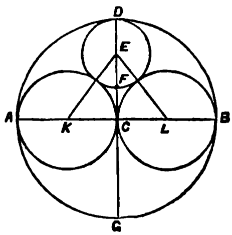
This solution does not assume that the points G, I, H, K lie in one plane.

**3572.** (Proposed by the Rev. W. H. LAVERTY, M.A.)—ACB is the diameter of a circle centre C; two equal confocal parabolas are drawn with their axes along AB; one touches the circle at A and is concave towards B, the other touches it at B and is concave towards A. An hyperbola is drawn with a focus at C, and with its axis along a line CDEF perpendicular to AB, and at its vertex touching the circle; also, the other branch of the hyperbola touches the parabolas; D is the point of contact of the circle and hyperbola, E is the foot of the nearer directrix of the hyperbola, F is the further vertex of the same curve. Show that CF = 2CE = 3CD.

I. *Solution by* A. M. NASH; *the* PROPOSER; *and others.*

Reciprocate with respect to C, and we get the following problem:—

If ACB be a diameter of a circle, and on AC, CB as diameters circles be drawn, and a fourth circle be drawn to touch the first circle internally, and the others externally; then if DEFCG be the diameter at right angles to ACB, CD will be trisected in E and F, the centre and circumference of the fourth circle.




Let K, L be the centres of the two equal circles; $r$, $\frac{1}{2}r$, $r_1$ the radii of the first, second, third, and fourth circles; then

$$KE^2 = KC^2 + CE^2,$$

or $$(\tfrac{1}{2}r + r_1)^2 = \tfrac{1}{4}r^2 + (r - r_1)^2;$$

therefore $r^2 = 3rr_1$, and $r = 3r_1$.

Hence CD is trisected in E and F; and therefore, by reciprocation, the given theorem is true.

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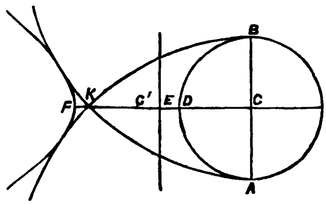
II. *Solution by* R. Tucker, M.A.; Dr. Hart; *and others.*




Let C′ be the centre of the hyperbola which we refer to axes C′C and a perpendicular thereto. Then if C′D $= a$, CD $= r$, we have the equations

$$b^2x^2-a^2y^2-a^2b^2=0 \quad\dots\dots\dots\dots(1$$

$$x^2-2(a+r)x+4ry+(a+r)^2-4r^2=0 \quad\dots\dots(2)$$

to the hyperbola and parabola respectively.

Since C is the focus of (1)

$$a^2+b^2=(a+r)^2 \;\text{ or }\; b^2=2ar+r^2 \quad\dots\dots\dots\dots\dots\dots\dots(3).$$

In order that (1) and (2) may touch, we have, using Salmon's notation,

$$\Delta=a^4b^4,\quad \Delta'=-4r^2,\quad \Theta=a^2b^2r(3r-2a),\quad \Theta'=4r^2(a^2-b^2),$$

to find the relation between $r$ and $a$, so that

$$\Delta k^3+\Theta k^2+\Theta' k+\Delta'=0 \quad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(4)$$

may have equal roots.

Making the above substitutions in (4), we get

$$a^4(2a+r)^2k^3+a^2(2a+r)(3r-2a)k^2+4(a^2-2ar-r^2)k-4=0.$$

Employing the condition for equal roots (Salmon's *Couics*, § 372), we have $(16a^3+16a^2r-4ar^2-3r^3)^2 =$

$$(21r^2+12ar-8a^2)(4r^4+16ar^3+17a^2r^2-4a^3r-8a^4),$$

which is clearly satisfied by $r=a$.

(The ultimate equation becomes

$$25r^6+120ar^5+199a^2r^4+72a^3r^3-160a^4r^2-192a^5r-64a^6=0).$$

Hence $\qquad$ CF $=$ 3CD, and since $\mathrm{C'E}=\dfrac{(\mathrm{C'D})^2}{\mathrm{C'C}}=\dfrac{a}{2}$,

therefore $\qquad$ CF $- 3a =$ 2CE.

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**3528.** (Proposed by Professor Wolstenholme.)—If

$$u=x_1(1-x_2)=x_2(1-x_3)=\dots=x_n(1-x_{n+1})=x_{n+1}(1-x_1),$$

and $x_1, x_2, \dots x_{n+1}$ be all unequal, then will $u$ be given by the equation

$$\left\{1+(1-4u)^{\frac12}\right\}^{n+1}=\left\{1-(1-4u)^{\frac12}\right\}^{n+1},$$

the factor $(1-4u)^{\frac12}$ being rejected.

*Solution by the* PROPOSER.

We have $x_2 = 1 - \frac{u}{x_1}$, $x_3 = 1 - \frac{u}{x_2} = 1 - \frac{u}{1-}\,\frac{u}{x_1}$, ......,

and finally $$x_1 = 1 - \frac{u}{1-}\,\frac{u}{1-}\,\cdots\,\frac{u}{-1-}\,\frac{u}{x_1},$$

there being $n$ units before $x_1$. Also, if we form the first three convergents, we have $\frac{1}{1}$, $\frac{1-u}{1}$, $\frac{1-2u}{1-u}$; and since the numerators and denominators are formed according to the same law $v_r = v_{r-1} - uv_{r-2}$ until the last, we see that the $r$th numerator will always be equal to the $(r+1)$th denominator. Also the $r$th denominator is

$$A\left(\frac{1+(1-4u)^{\frac{1}{2}}}{2}\right)^{r-1} + B\left(\frac{1-(1-4u)^{\frac{1}{2}}}{2}\right)^{r-1},$$

where $$A + B = 1,\quad A\,\frac{1+(1-4u)^{\frac{1}{2}}}{2} + B\,\frac{1-(1-4u)^{\frac{1}{2}}}{2} = 1;$$

therefore $$A = \frac{1+(1-4u)^{\frac{1}{2}}}{2(1-4u)^{\frac{1}{2}}},\quad B = \frac{-1+(1-4u)^{\frac{1}{2}}}{2(1-4u)^{\frac{1}{2}}};$$

therefore the $r$th denominator is

$$\frac{1}{(1-4u)^{\frac{1}{2}}}\left\{\left(\frac{1+(1-4u)^{\frac{1}{2}}}{2}\right)^{r} - \left(\frac{1-(1-4u)^{\frac{1}{2}}}{2}\right)^{r}\right\}.$$

We shall then have the equation for the last convergent

$$x_1 = \frac{x_1 p_{n+1} - up_n}{x_1 q_{n+1} - uq_n} = \frac{x_1 q_{n+2} - uq_{n+1}}{x_1 q_{n+1} - uq_n}.$$

Now it is evident that the equations will all be satisfied, whatever be the value of $u$, if we have $x_1^2 - x_1 + u = 0$, which gives $x_2 = x_1$, $x_3 = x_2$, .... This quadratic expression in $x_1$ is then a factor of

$$x_1^2 q_{n+1} - x_1(uq_n + q_{n+2}) + uq_{n+1},$$

and the other factor is $q_{n+1}$ (as is otherwise obvious, since $q_{n+1} - q_{n+2} \equiv uq_n$); and since it is given that $x_1, x_2, \ldots$ are unequal, we must have $q_{n+1} = 0$,

or $$\frac{1}{(1-4u)^{\frac{1}{2}}}\left\{\left(\frac{1+(1-4u)^{\frac{1}{2}}}{2}\right)^{n+1} - \left(\frac{1-(1-4u)^{\frac{1}{2}}}{2}\right)^{n+1}\right\} = 0,$$

or $$\left\{1+(1-4u)^{\frac{1}{2}}\right\}^{n+1} = \left\{1-(1-4u)^{\frac{1}{2}}\right\}^{n+1},$$

if we reject the factor $(1-4u)^{\frac{1}{2}}$. If we put

$$1 - 4u = -\tan^2\theta,\quad \text{or}\quad u = \tfrac{1}{4}\sec^2\theta,$$

we have $\sin(n+1)\theta = 0$, or $\theta = \frac{r\pi}{n+1}$, and $u = \frac{1}{4}\sec^2\frac{r\pi}{n+1}$,

giving $\frac{1}{2}(n-1)$ or $\frac{1}{2}n$ different values of $u$ according as $n$ is odd or even. The values of $r$ range from 1 to $n$, but the values of $u$ are the same for two values of $r$ whose sum is $n+1$. If $u$ have any value not included in this system, the given equations can only be satisfied by

$$x_1 = x_2 = \ldots = x_{n+1} = \tfrac{1}{2}\left\{1 \pm (1-4u)^{\frac{1}{2}}\right\}.$$

If $u$ have one of these values, the given equations are not independent, and the values of $x_1$ ... are indeterminate.

**3244.** (Proposed by Rev. A. F. TORRY, M.A.)—Through any point P, within or without a parabola whose focus is S, a double ordinate is drawn; the polar of P cuts the axis in M; the perpendicular from P upon this polar meets it in N and the axis in R; show that M, N, Q, R, Q′ all lie on a circle whose centre is S.

*Solution by* STEPHEN WATSON.

Let $(h, k)$ denote the point P, the focus being the origin; then the equations of the parabola, the polar of P, and a perpendicular to this polar from P, are $y^2 = 4m(m+x)$, $ky - 2mx = 2m(h+2m)$ ............ (1, 2),

$$2my + kx = k(h+2m) \quad \text{.........................} \quad (3);$$

hence if $x_1$, $y_1$ be the coordinates of N, we have

$$(k^2+4m^2)\,x_1 = (k^2-4m^2)(h+2m), \quad (k^2+4m^2)\,y_1 = 4mk(h+2m);$$

therefore $\qquad SN = (x_1^2+y_1^2)^{\frac{1}{2}} = h+2m.$

Also putting $y=0$ in (2) and (3), we have

$$SM = SR = h+2m,$$

and it is well known that $\quad SQ = SQ' = h+2m;$

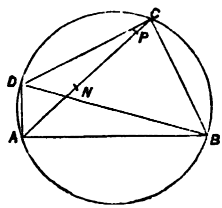
therefore $\qquad SM = SN = SR = SQ = SQ'.$

[A Solution by Dr. HIRST is given on p. 83 of Vol. XIV. of the *Reprint*.]

**3523.** (Proposed by R. TUCKER, M.A.)—B, C, D are three points on an ellipse, the osculating circles at which cointersect at A, also on the curve. If O, O′ be the centres of the ellipse and rectangular hyperbola through A, B, C, D, then the ortho-centres of the four triangles that can be formed from the four points lie not only on the hyperbola, but are situated on an ellipse whose centre lies on OO′ at a distance of 2OO′ from O, analogously to the points A, B, C, D on the original ellipse.

I. *Solution by* R. W. GENESE, B.A.

This theorem is true for any four points of the ellipse which lie on a circle. In fact, if A, B, C, D lie on a circle, the orthocentres of the triangles BCD, CDA, &c. are the angular points of a quadrilateral exactly equal to ABCD.




Let P be the orthocentre of the triangle BCD. Then the nine-point circle is the locus of the middle points of radii from P to the circle BCD. Thus N, the middle point of PA, is on the nine-point circle, and can therefore be the centre of an equilateral hyperbola passing through BCD. Since P is on this curve, A must be also, or N is the centre of the equilateral hyperbola ABCD.

Obviously the other orthocentres may be found by producing BN, CN, DN their own lengths. The theorem follows immediately.

II. *Solution by the* PROPOSER.

The equation to the rectangular hyperbola may be readily found by the method employed in my solution of Quest. 2843 (*Reprint*, Vol. XII. p. 45), or at once got from the equations to the parabolas given on p. 44. It is

$$y^2-x^2+x\cos\delta\,\frac{a^2+b^2}{2a}-y\sin\delta\,\frac{a^2+b^2}{2b}+\frac{a^2-b^2}{2}=0 \;\ldots\ldots (1);$$

whence the coordinates of O′ are $\dfrac{a^2+b^2}{4a}\cos\delta,\ \dfrac{a^2+b^2}{4a}\sin\delta.$

If A $(\delta)$, B $(-\tfrac{1}{3}\delta)$, C $(120°-\tfrac{1}{3}\delta)$, D $(240°-\tfrac{1}{3}\delta)$ are the points on the ellipse, then the ortho-centre of BCD is

$$-\frac{a^2-b^2}{2a}\cos\delta,\quad \frac{a^2-b^2}{2b}\sin\delta;$$

of ABC is $\quad a\sin(30°+\tfrac{1}{3}\delta)+\dfrac{a^2+b^2}{2a}\cos\delta,\quad b\cos(30°+\tfrac{1}{3}\delta)+\dfrac{a^2+b^2}{2b}\sin\delta;$

of ABD is

$$a\sin(30°-\tfrac{1}{3}\delta)+\frac{a^2+b^2}{2a}\cos\delta,\quad -b\sin(30°-\tfrac{1}{3}\delta)+\frac{a^2+b^2}{2b}\sin\delta;$$

of ACD is $\quad -a\cos\tfrac{1}{3}\delta+\dfrac{a^2+b^2}{2a}\cos\delta,\quad b\sin\tfrac{1}{3}\delta+\dfrac{a^2+b^2}{2b}\sin\delta.$

It may be verified by substitution that these four points lie (as is well known) on (1).

Take a point O″ on OO′ produced so that OO″ = 2OO′, then its coordinates will be $\qquad \dfrac{a^2+b^2}{2a}\cos\delta,\quad \dfrac{a^2+b^2}{2b}\sin\delta;$

and now transfer the origin to O″, and the coordinates of the four ortho-centres become, when referred to axes through O″ parallel to the original axes, $\ -a\cos\delta,\ -b\sin\delta;\quad a\sin(30°+\tfrac{1}{3}\delta),\ b\cos(30°+\tfrac{1}{3}\delta);$

$$a\sin(30°-\tfrac{1}{3}\delta),\ -b\cos(30°-\tfrac{1}{3}\delta);\quad -a\cos\tfrac{1}{3}\delta,\ b\sin\tfrac{1}{3}\delta.$$

These values show that the ortho-centres are similarly situated on an equal and similarly situated ellipse; *i.e.*, the osculating circles at the last three cointersect at the first point, eccentric angle $\pi-\delta$.

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**3587.** (Proposed by R. TUCKER, M.A.)—The ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{b^2x^2}{a^2}+\frac{a^2y}{b^2}=a^2+b^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (1, 2)$$

are so related that if tangents be drawn from any point on (2) to (1), meeting it, say, in Q, and circles concentric with the ellipses be described with radius CQ, &c.; then if common tangents to these circles and (1) be drawn, the ordinate of the point of contact P, on (1) produced, meets CQ produced on the auxiliary circle of (1).

I. *Solution by the* Rev. W. H. Laverty, M.A.

This property is part of a more general one, being independent, in fact, of ellipse (2), and true wherever Q may be on ellipse (1). To show this: with centre C and radius CQ draw a circle. Produce CQ to meet the auxiliary circle in T. Draw the ordinate TPN meeting the ellipse. Then the tangent at P will touch the circle through Q, as may be thus shown:—

Let the coordinates of Q be $-\dfrac{ab}{(b^2+a^2m^2)^{\frac{1}{2}}}, \quad \dfrac{mab}{(b^2+a^2m^2)^{\frac{1}{2}}}$;

therefore „ „ T are $-\dfrac{a}{(1+m^2)^{\frac{1}{2}}}, \quad \dfrac{ma}{(1+m^2)^{\frac{1}{2}}}$;

and „ „ P are $-\dfrac{a}{(1+m^2)^{\frac{1}{2}}}, \quad \dfrac{mb}{(1+m^2)^{\frac{1}{2}}}$;

the tangent at P may be put into the form

$$-x \,.\, \frac{ab^2(1+m^2)^{\frac{1}{2}}}{b^2+a^2m^2} + y \,.\, \frac{ma^2(1+m^2)^{\frac{1}{2}}}{b^2+a^2m^2} = \frac{a^2b^2(1+m^2)}{b^2+a^2m^2};$$

therefore it touches the circle through Q.

II. *Solution by the* Proposer.

The ellipse (2) is the locus of the intersection of tangents drawn to (1) at the extremities of perpendicular central vectors. [I was not aware, till subsequently, that this exercise is given in Todhunter's *Conics*, Ch. ix. Ex. 37, but now use the property as it shortens my proof.]

Then if P, P′ be the points on the ellipse whose eccentric angles are the angles which the above central vectors make with the major axis, it is clear that CP, CP′ are semi-conjugate diameters, and the tangents at P, P′ will be touched by circles whose radii are CQ, CQ′ respectively, for

$$CQ^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} \quad \text{(Todhunter, § 198)}$$

$$= (\text{perpendicular from centre on tangent at P})^2;$$

whence the truth of the question.

**3505.** (Proposed by J. B. Sanders.)—A body, projected in a direction making an angle $\alpha$ with a plane whose inclination to the horizon is $\beta$, fell upon the plane at the distance $a$ from the point of projection, which is also in the inclined plane. Required the velocity of projection and the time of flight.

*Solution by* Artemas Martin.

Putting $y = x\tan\beta$ in the equation of the trajectory, viz.,

$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}, \quad \text{we obtain} \quad x = \frac{2v^2\cos\alpha\sin(\alpha-\beta)}{g\cos\beta};$$

therefore $$a = \frac{x}{\cos\beta} = \frac{2v^2 \cos\alpha \sin(\alpha-\beta)}{g\cos^2\beta};$$

whence $$v = \left(\frac{ag\cos^2\beta}{2\cos\alpha\sin(\alpha-\beta)}\right)^{\frac{1}{2}},$$

and $$t = \frac{2v\sin(\alpha-\beta)}{g\cos\beta} = \left(\frac{2a\sin(\alpha-\beta)}{g\cos\alpha}\right)^{\frac{1}{2}}.$$

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**3542.** (Proposed by G. O'Hanlon.)—Find the normal which cuts off the least curve from a conic.

*Solution by* J. J. Walker, M.A.

Taking first the case of a central conic referred to its axes $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right)$, let PQ be the required normal (at P) chord; then, if P′ be the consecutive point to P on the curve, and P′Q′ the normal (at P′) chord, intersecting PQ in O; and QR, parallel to PP′, meet P′Q′ in R, we shall have $\frac{OQ}{OP}=\frac{QR}{PP'}=\frac{QR}{QQ'}=\sin\phi$, where $\phi$ is the angle which the tangent at Q makes with PQ; or writing PQ − OP for OQ, squaring and reducing,

$$\frac{PQ^2}{OP^2}-2\frac{PQ}{OP}+\cos^2\phi = 0.$$

If P be $(x, y)$ and Q be $(x', y')$, it is easily found that

$$x' = -x + \frac{2a^4c^2xy^2}{b^6x^2+a^6y^2},\quad y' = -y - \frac{2b^4p^2x^2y}{b^6x^2+a^6y^2};$$

and, recollecting that OP is the radius of curvature at P,

$$\frac{PQ}{OP} = \frac{2a^4b^4}{b^6x^2+a^6y^2},\quad \tan^2\phi = \frac{(b^6x^2+a^6y^2)^2}{4a^4b^4c^4x^2y^2}.$$

Consequently the equation above becomes, after division by $4a^4b^4$,

$$\frac{a^4b^4}{(b^6x^2+a^6y^2)^2} - \frac{1}{b^6x^2+a^6y^2} + \frac{c^4x^2y^2}{(b^6x^2+a^6y^2)^2+4a^4b^4c^4x^2y^2} = 0,$$

which, cleared of fractions, and made homogeneous in $x^2$, $y^2$ by means of the equation to the conic, gives, to determine $\frac{x^2}{y^2}$,

$$\left\{(b^6x^2+a^6y^2)^2+4a^4b^4c^4x^2y^2\right\}(b^6x^4-a^6y^4)-4a^6b^6c^6x^4y^4 = 0,$$

or, after development,

$$b^{18}x^8+2a^4b^{10}(a^2b^2+2c^4)x^6y^2+a^6b^6(a^6-b^6-4c^6)x^4y^4$$
$$-2a^{10}b^4(a^2b^2+2c^4)x^2y^6-a^{18}y^8 = 0,$$

which has one, and only one, positive real root, the signs of the extreme

terms being essentially opposite, and there being only one alternation of signs among the coefficients, whether that of $x^4y^4$ be essentially positive or negative.

Writing $a-x$ for $x$ in the last form but one, dividing by $a^{18}b^4$, putting $\frac{4b^2}{a}=p$, lastly making $a$ and $b$ infinite, we get for the parabola $y^2-px=0$, $x=\frac{1}{2}p$ as the point at which the normal chord cuts off the least arc; or this result may be obtained independently from $\frac{2\text{PQ}}{\text{OP}}=\frac{p}{x}$, $\tan^2\phi=\frac{x^2}{y^2}$.

The above method is shorter than the more purely analytical one of equating with zero the differential with respect to $x$ of $\int_{x'}^{x}\left(1+\frac{dy^2}{dx^2}\right)dx$,

where $y$ is given in terms of $x$ by $\frac{x^2}{b^2}+\frac{y^2}{a^2}-1=0$,

and
$$x'=-x+\frac{2a^4(a^2-b^2)\,xy^2}{b^6x^2+a^6y^2},$$
which would lead to the same result.

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**3496.** (Proposed by S. Bills.)—To find three positive whole numbers such that their sum, the sum of their squares, and the sum of their cubes shall be rational squares.

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I. *Solution by* Asher B. Evans, M.A.

Let $ax$, $ay$, $az$ represent the three numbers; then we must have
$$a(x+y+z)=\square,\quad a^2(x^2+y^2+z^2)=\square,\quad a^3(x^3+y^3+z^3)=\square\ \ldots(1,2,3),$$
(1) is satisfied by putting $a=x+y+z$, and (2) and (3) reduce to
$$x^2+y^2+z^2=\square,\quad (x+y+z)(x^3+y^3+z^3)=\square\ \ldots\ldots\ldots\ (4,5).$$
Let $(x+y+z)(x-y+z)$ be the root of (5); then we have
$$(x+y+z)(x^3+y^3+z^3)=(x+y+z)^2(x-y+z)^2,$$
whence $\quad y^2+y(x+z)=3xz,\quad$ or $\quad 2y=(x^2+14xz+z^2)^{\frac{1}{2}}-x-z\ \ldots\ldots\ldots\ (6).$

In order to satisfy (6), we must make $\quad x^2+14xz+z^2=\square\ \ldots\ldots\ldots\ldots\ (7).$

Put $\left(\frac{m}{n}z-y\right)$ for the root of (7); then $\frac{z}{x}=\frac{2mn+14n^2}{m^2-n^2}$.

Take $z=2mn+14n^2$, and $x=m^2-n^2$; then $y=6mn-6n^2$.

By substituting these values of $x$, $y$, $z$ in (4), we obtain
$$x^2+y^2+z^2=m^4+38m^2n^2-16mn^3+233n^4\ \ldots\ldots\ldots\ldots\ldots\ldots\ (8),$$
which, put equal to $(m^2+19n^2)^2$, gives $m=-8n$. Let $m=p-8n$, and put (8) equal to $(p^2-16pn-83n^2)^2$; then $\frac{p}{n}=\frac{1332}{83}$. Take $p=1332$; then $n=83$, $m=668$, $x=439335$, $y=291330$, $z=207334$, $a=937999$,
$$ax=412095790665,\quad ay=273267248670,\quad az=194479084666.$$

II. *Solution by the* Proposer; A. Martin; Dr. Hart; *and others.*

Let $15mx$, $15my$, and $8m(x+y)$ denote the three numbers, then we must have $\quad 23m(x+y) = \square, \quad 225m^2(x^2+y^2) + 64(x+y)^2 = \square;$

and $$15^3m^3(x^3+y^3) + 8^3m^3(x+y)^3 = \square;$$

or, dividing the last of these by the first, and reducing, we have to find

$$23m(x+y) = \square, \quad x^2+axy+y^2 = \square, \quad \text{and} \quad x^2-bxy+y^2 = \square \dots (1, 2, 3);$$

where $$a = \tfrac{128}{289}, \text{ and } b = \tfrac{2351}{3887}.$$

If we take $x = p^2-q^2$ and $y = 2pq+aq^2$, (2) will be satisfied, and substituting these values in (3), it will become

$$p^4 - 2bp^3q + (2-ab)p^2q^2 + 2(2a+b)pq^3 + (a^2+ab+1)q^4 = \square \;\dots\dots (4).$$

Assume $$(4) = \left\{p^2 - bpq - \tfrac{1}{2}(b^2+ab-2)q^2\right\}$$

then we find $p = -\frac{1}{4}(a+b)q$.

Taking $q=4$, we have $p = -(a+b)$; whence we find

$$x = (a+b+4)(a+b-4), \quad \text{and} \quad y = 8(a-b);$$

or, changing the signs, which we are at liberty to do, we have

$$x = (4+a+b)\left\{4-(a+b)\right\}, \quad \text{and} \quad y = 8(b-a),$$

which, in the present example, will be found to be both positive.

Lastly assume $23m(x+y) = d^2$, then $m = \dfrac{d^2}{23(x+y)}$, and the three numbers will be $\quad \dfrac{15d^2x}{23(x+y)}, \quad \dfrac{15d^2y}{23(x+y)}, \text{ and } \dfrac{8d^2}{23}.$

It is evident that, by taking a suitable value for $d$, these expressions may be made all *integral.*

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**3493.** (Proposed by W. H. H. Hudson, M.A.)—Towards the end of a paper a man has two questions to do in $t$ minutes; their difficulties are as $d_1 : d_2$, and the marks for them as $m_1 : m_2$; his chance of doing either varies as the square of the time he spends upon it, and inversely as its difficulty. How may he most profitably divide his time between them?

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*Solution by* G. S. Carr.

If the marks gained for an unfinished question be represented by the expectation; and if the expectations be respectively $\dfrac{x_1^2m_1}{d_1}$ and $\dfrac{x_2^2m_2}{d_2}$, where $x_1$, $x_2$ are the times spent upon the questions; then it is required to find $x_1$, $x_2$, so that $\dfrac{x_1^2m_1}{d_1} + \dfrac{x_2^2m_2}{d_2}$ may be a maximum, while $x_1+x_2 = t$.

By differentiating we have for this condition, $\dfrac{2x_1m_1}{d_1} = \dfrac{2x_2m_2}{d_2}$; therefore

$$x_1 = \frac{m_2d_1t}{m_1d_2+m_2d_1}, \quad x_2 = \frac{m_1d_2t}{m_1d_2+m_2d_1}.$$

**3549.** (Proposed by J. J. WALKER, M.A.)—If the coordinates of any point not on the curve are substituted in the expression which, equated to zero, is the Cartesian equation of a central conic, show that the result is proportional to the square of the area of the quadrilateral formed by tangents from that point and semi-diameters to the points of contact.

I. *Solution by* STEPHEN WATSON.

Let $x' = a\cos\theta,\ y' = b\sin\theta,\ x'' = a\cos\theta_1,\ y'' = b\sin\theta_1$ be the points of contact; then the point of intersection of the tangents is

$$x_1 = \frac{a^2(y'-y'')}{x''y'-x'y''},\quad y_1 = \frac{b^2(x'-x'')}{x'y''-x''y'} \quad \dots\dots\dots\dots\dots\dots (1),$$

and the area of the quadrilateral is

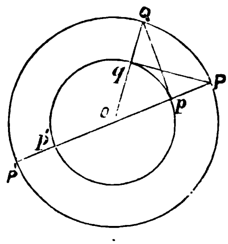
$$\tfrac{1}{2}\left\{(y_1+y')(x'-x_1)+(y_1+y'')(x_1-x'')-x'y'+x''y''\right\}$$
$$=\tfrac{1}{2}\left\{y_1(x'-x'')-x_1(y'-y'')\right\} = \frac{b^2(x'-x'')^2+a^2(y'-y'')^2}{2(x'y''-x''y')}$$
$$= ab\tan\tfrac{1}{2}(\theta_1-\theta).$$

Again, putting the values of $x_1, y_1$, above, in $a^2b^2\left(\dfrac{y_1^2}{b^2}+\dfrac{x_1^2}{a^2}-1\right)$, the result easily reduces to $a^2b^2\tan^2\frac{1}{2}(\theta_1-\theta) = (\text{area of quadrilateral})^2$.

The above solution is for the ellipse, but the final result obviously holds good in the case of the hyperbola.

Similarly, in the case of the parabola $y^2-px=0$, $y_1^2-px=\frac{1}{4}(y'-y'')^2$, as it should be from the result in the case of the ellipse.

II. *Solution by* Rev. G. H. HOPKINS, M.A.




Let P be the point which, projected upon the plane of the conic, has for its coordinates $x, y$; and O be the centre of the circle $pqp'$ which can be projected into the given conic.

Then $\qquad \mathrm{PO}^2 = x^2+y^2\cdot\dfrac{a^2}{b^2}$;

therefore $(\mathrm{PO}^2-a^2)\,b^2 = b^2x^2+a^2y^2-a^2b^2$;

therefore $b^2x^2+a^2y^2-a^2b^2 = p'p\,.\,p\mathrm{P}\,.\,a^2\,.\,\dfrac{b^2}{a^2}$

$$= \mathrm{Q}p^2\,.\,\mathrm{O}p^2\,.\,\frac{b^2}{a^2} = (2\text{ area of }\mathrm{QO}p)^2\,.\,\frac{b^2}{a^2}$$

$= (2\text{ area of }q\mathrm{OP})^2\,.\,\dfrac{b^2}{a^2} = (2\text{ area of the projection of }\mathrm{P}q\mathrm{O})^2$

= the square of the area of the quadrilateral formed by the tangents from the projection of $p$ and the semi-diameters to the points of contact.

**3491.** (Proposed by C. TAYLOR, M.A.)—A sphere and a luminous point move about in any manner in space. Determine the relation between their motions when the least diameter of the shadow of the sphere cast on a fixed plane is constant.

*Solution by* G. S. CARR.

Let A be the luminous point, B the centre of the sphere; and let two tangents to the sphere from A, drawn in a vertical plane through A and B, cut a fixed horizontal plane in C and D.

Then the shadow is an ellipse having CD for its major axis, and its minor equal to twice the mean proportional between the perpendiculars from C and D upon AB. Therefore the sphere and the point must so move that $AC \cdot AD \sin^2 BAC$ may be constant.

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**3612.** (Proposed by T. MITCHESON, B.A.)—Two parallel straight lines are drawn, terminating in the middle points of four sides of a regular pentagon. Prove that the radius of the inscribed circle is equal to twice the difference of the perpendiculars on the parallel lines from the centre of this circle.

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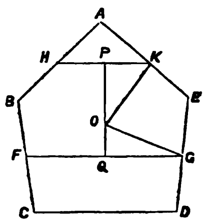
*Solution by* Q. W. KING; J. H. SLADE; *the* PROPOSER; *and others.*




Let HK and FG be the two parallel straight lines, and O the centre of the inscribed circle. Draw OP perpendicular to HK, and OQ perpendicular to FG. Then OP and OQ are in the same straight line.

Now $\angle PKO = 54°$, and $\angle OGQ = 18°$;

therefore $OP - OQ = OG(\sin 54° - \sin 18°)$

$$= OG\left\{\tfrac{1}{4}(\sqrt{5}+1) - \tfrac{1}{4}(\sqrt{5}-1)\right\} = \tfrac{1}{2}OG.$$

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**3590.** (Proposed by Dr. SYLVESTER.)—In the game of bowls, whichever side, after all have thrown, has one or more balls nearer to the Jack than any of the balls of the opposite side, counts so many points towards the game. Construct the adjoining triangle of polygonal numbers. Required to show, that if each side plays with $n$ balls, the chance, after all have thrown, of one side or the other making $i$ points is represented by a fraction whose numerator is the $i$th figure (reckoning from right to left) in the $n$th horizontal line of the triangle, and its denominator the sum of all the figures in that line, or the figure immediately below its last term in the following line.

1
1, 2
1, 3, 6
1, 4, 10, 20
1, 5, 15, 35, 70
... ... ... ... ... ...
... ... ... ... ... ... ...
... ... ... ... ... ... ... ...

*Solution by* ANTHONY TRAILL, LL.D., M.D.

Let A on one side have $n$ bowls ......... o o o o o o ......,

and B on the other side have $n$ bowls ... x x x x x x .......

If A make $i$ points, it is necessary to have $i$ balls o o o o ...... followed by one ball x, all the remaining balls lying in any order. But for $i$ particular balls o o o o ...... followed by one particular ball x, the number of possible ways in which this can happen is represented by the number of permutations of the remaining $(2n-i-1)$ balls, *i.e.* by the number $1.2.3.4\ldots(2n-i-1)$ or $\underline{|2n-i-1}$. But if we remove the restrictions on the particular $i$ balls o o o o and the particular ball x, we can have $n(n-1)(n-2)\ldots(n-i+1)$ groups of $i$ balls o o o o ......, and $n$ balls x; therefore the total number of ways in which A can score $i$ points is

$$n(n-1)(n-2)\ldots(n-i+1).n.\underline{|2n-i-1}.$$

But B can score $i$ points in an equal number of ways; therefore the total number of ways in which $i$ points can be scored by either side is

$$2n.n(n-1)(n-2)\ldots(n-i+1).\underline{|2n-i-1}.$$

But the total number of ways in which all the balls may lie in any order is equal to the number of permutations of $2n$ balls, *i.e.* $\underline{|2n}$; therefore the chance required is given by the fraction

$$\frac{2n.n(n-1)(n-2)\ldots(n-i+1).\underline{|2n-i-1}}{2n(2n-1)(2n-2)\ldots3.2.1}, \quad\text{or}\quad \frac{\underline{|n}.\underline{|2n-i-1}}{\underline{|2n-1}.\underline{|n-i}} \ldots (\alpha).$$

But the $n$th row in the table given above consists of $n$ terms

$$1,\ n,\ \frac{n(n+1)}{1.2},\ \frac{n(n+1)(n+2)}{1.2.3},\ \ldots\ldots,$$

of which the $i$th term from the end, or $(n-i+1)$th term from the beginning, is
$$\frac{n(n+1)(n+2)\ldots(2n-i-1)}{1.2.3\ldots(n-i)},$$

and the sum of all the figures in the $n$th row is the same as the $n$th figure in the $(n+1)$st row, *i.e.* $\dfrac{(n+1)(n+2)\ldots(2n-1)}{\underline{|n-1}}$,

whence the fraction given by Dr. Sylvester is

$$\frac{n(n+1)(n+2)\ldots(2n-i-1)}{\underline{|n-i}}.\frac{\underline{|n-1}}{(n+1)(n+2)\ldots(2n-1)},$$

or
$$\frac{\underline{|2n-i-1}}{\underline{|n-i}}.\frac{\underline{|n}}{\underline{|2n-1}} \ldots\ldots\ldots\ldots\ldots\ldots (\beta).$$

But the chance calculated above $(\alpha)$ has the same value as $(\beta)$; therefore, &c.

[The Proposer remarks that "bowls" is still a favourite game in Scotland, and the question suggested itself to him in consequence of his seeing it played in the cool of the evening at the beautifully-kept bowling-green in Oban, Argyleshire, which he was most courteously invited to enter in the course of a visit to Scotland last autumn.]

**3591.** (Proposed by Professor CAYLEY.)—If in a plane A, B, C, D are fixed points and P a variable point, find the linear relation

$$\alpha \, . \, PAB + \beta \, . \, PBC + \gamma \, . \, PCD + \delta \, . \, PDA = 0$$

which connects the areas of the triangles PAB, &c.

*Solution by* Professor TOWNSEND, M.A., F.R.S.

The relation in question, holding for every point P in the plane, holds consequently for the four points A, B, C, D. Hence, if O be the intersection of the two diagonals AC and BD of the quadrilateral ABCD, we have, to determine the several ratios $\alpha : \beta : \gamma : \delta$, the four relations

$$\left.\begin{aligned} \beta \, . \, BAC + \gamma \, . \, CAD &= 0, \\ \gamma \, . \, CBD + \delta \, . \, DBA &= 0, \\ \delta \, . \, DCA + \alpha \, . \, ACB &= 0, \\ \alpha \, . \, ADB + \beta \, . \, BDC &= 0, \end{aligned}\right\}, \text{ or } \left\{\begin{aligned} \beta \, . \, BOC + \gamma \, . \, COD &= 0, \\ \gamma \, . \, COD + \delta \, . \, DOA &= 0, \\ \delta \, . \, DOA + \alpha \, . \, AOB &= 0, \\ \alpha \, . \, AOB + \beta \, . \, BOC &= 0; \end{aligned}\right.$$

from which it appears at once that

$$\alpha \, . \, AOB = -\beta \, . \, BOC = \gamma \, . \, COD = -\delta \, . \, DOA,$$

and therefore that $\dfrac{APB}{AOB} - \dfrac{BPC}{BOC} + \dfrac{CPD}{COD} - \dfrac{DPA}{DOA} = 0,$

which is consequently the required relation.

**3593.** (Proposed by the Rev. Dr. BOOTH, F.R.S.)—Two parallel tangents to a plane section of a right cone meet in the points M, $M_1$ a tangent plane to the cone which touches the focal spheres in the points Q, $Q_1$. Show that $MQM_1Q_1$ is a quadrilateral which may be inscribed in a circle.

*Solution by* C. W. MERRIFIELD, F.R.S.; Professor WOLSTENHOLME; *and others.*

Using the figure by the help of which the focal property of the section of the cone is commonly proved, it is easily seen that the property stated depends upon the proposition that, if a pair of parallel tangents to a plane conic meet a fixed tangent, the rectangle contained by the segments of the fixed tangent is equal to that contained by the focal distances of the point.

Since involution is a projective property, this is at once proved by projection from the circle. The rectangle is then seen to be equal to the square on the semi-diameter conjugate to the point on the ellipse, and therefore to the rectangle between the focal distances of the point.

**3595.** (Proposed by Professor CROFTON, F.R.S.)—If A be the first point of a cycloid, P any position of the describing point, T the lowest point of the rolling circle; a perpendicular from T on the chord AP passes through the centre of gravity of the arc PT.

I. *Solution by* Professor TOWNSEND, M.A., F.R.S.

This property is manifestly equivalent to the following:—

If AB be any arc of a circle, C its centre of gravity, and AD the tangent of equal length at either of its extremities A; the line AC is perpendicular to the line BD.

This may be easily proved as follows:—Drawing OA, OC, and AB, where O is the centre of the circle; then since, by a well-known property, OC : OA = AB : AD, and since, from the circle, OC and OA are perpendicular to AB and AD; therefore, from the two similar triangles COA and BAD, AC is perpendicular to BD.

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II. *Solution by* J. J. WALKER, M.A.

Let O be the centre of the generating circle, and let OG, perpendicular to the chord PT, meet the perpendicular from T on AP in G. The triangle OGT has its sides perpendicular respectively to those of the triangle TPA; consequently these two triangles are equiangular, and TA : TP = OT : OG; *i.e.*, OG is a fourth proportional to the circular arc, its chord and the radius, and is perpendicular to the chord; therefore G is the centre of gravity of the arc.

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**3644.** (Proposed by T. MITCHESON, B.A.)—Solve the equation

$$x^6+6x^5+17x^4+9x^3-29x^2-60x+56=0.$$

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I. *Solution by* HUGH MCCOLL.

It is evident by inspection that 1 is a root of the equation, so that it resolves into the form $(x-1)(x^5+7x^4+24x^3+33x^2+4x-56)=0$.

The expression in the second bracket may be resolved by a very simple and elementary process (see my *Algebraical Exercises*) into the form $(x^2+4x+8)(x^3+3x^2+4x-7)$. Putting, as usual, $i$ for $\sqrt{(-1)}$, the roots of the quadratic are $-2\pm 2i$; and the roots of the cubic are approximately

$$\cdot 920175121347 \quad \text{and} \quad -1\cdot 960087560674\pm 1\cdot 940439221537i.$$

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II. *Solution by* A. M. NASH; A. MARTIN; *the* PROPOSER; *and others.*

Perform the operation of extracting the square root, and take the first three terms of the result; then the proposed equation may be written in the quadratic form $(x^3+3x^2+4x)^2-15(x^3+3x^2+4x)=-56$;

whence $$x^3+3x^2+4x=8 \text{ or } 7.$$

The only real root of $x^3+3x^2+4x=8$, is 1.

The only real root of $x^3+3x^2+4x=7$, as found by Cardan's Rule, is

$$\left\{\frac{9}{2}+\left(\frac{81}{4}+\frac{1}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}+\left\{\frac{9}{2}-\left(\frac{81}{4}+\frac{1}{27}\right)^{\frac{1}{2}}\right\}-1.$$

**3566.** (Proposed by Professor WOLSTENHOLME.)—If $x$, $y$ be two real quantities connected by the equation $x^2+y^2-xy-y+\frac{1}{4}=0$; prove that $0$, $x$, $y$, $1$ will be in order of magnitude.

*Solution by the* Rev. W. H. LAVERTY, M.A.; R. TUCKER, M.A.; *and others.*

1. Putting the equation in the form $y^2-y(x+1)+(x^2+\frac{1}{4})=0$, we must have $(x+1)^2>4(x^2+\frac{1}{4})$, therefore $x(x-\frac{2}{3})<0$, therefore $x>0<\frac{2}{3}$.

2. Putting the equation in the form $x^2-xy+(y^2-y+\frac{1}{4})=0$, we get $y^2>4(y^2-y+\frac{1}{4})$, therefore $(y-1)(y-\frac{1}{3})<0$, therefore $y>\frac{1}{3}<1$.

3. Let $y=kx$, then $x^2(1-k+k^2)-kx+\frac{1}{4}=0$, $k^2>1-k+k^2$; therefore $k>1$; therefore $y>x$, and both $y$ and $x>0<1$.

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**3555.** (Proposed by Dr. MATTESON.)—Find $x$, $y$, $z$ so that

$$x^2+y^2\pm a(x+y)=\square,\quad y^2+z^2\pm b(y+z)=\square,\quad z^2+x^2\pm c(z+x)=\square,$$

$a$, $b$, $c$ having any values, and the sign $\pm$ being taken *disjunctively.*

*Solution by* DR. HART.

$$x^2+y^2+a(x+y)=\square=(x+y)^2,\quad\text{whence}\quad a(x+y)=2xy\ \ldots\ldots\ (1);$$
$$x^2+z^2+b(x+z)=\square=(x+z)^2,\quad\text{whence}\quad b(x+z)=2xz\ \ldots\ldots\ (2);$$
$$y^2+z^2+c(y+z)=\square=(y+z)^2,\quad\text{whence}\quad c(y+z)=2yz\ \ldots\ldots\ (3).$$

If we put the same expressions, with the negative sign, equal respectively to $(x-y)^2$, $(x-z)^2$, $(y-z)^2$, we shall have the same results as above, after changing the signs. We have then only to solve equations (1), (2), (3), to satisfy both the positive and negative conditions of the problem.

From (1), (2) we have $\quad x=\dfrac{ay}{2y-a}=\dfrac{bz}{2z-b},\quad$ whence

$$y=\frac{abz}{2bz-2az+ab}=\frac{cz}{2z-c},\ \text{by (3)};\quad\text{therefore}\quad z=\frac{abc}{ab+ac-bc};$$

also $\quad y=\dfrac{cz}{2z-c}=\dfrac{abc}{ab+bc-ac},\quad$ and $\quad x=\dfrac{ay}{2y-a}=\dfrac{abc}{ac+bc-ab}.$

In these general values of $x$, $y$, $z$, we must take $a$, $b$, $c$ so that the sum of any two of the products $ab$, $ac$, $bc$ shall be greater than the third.

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**3574.** (Proposed by T. COTTERILL, M.A.)—Show that the intercept of the tangent to a circle of radius $k$ between its asymptotes is constant and equal to $2k\sqrt{(-1)}$.

*Solutions by* A. B. EVANS, M.A.; R. W. GENESE, B.A.; Prof. TOWNSEND; *and others.*

1. This is a particular case of the theorem that any tangent to one of two concentric circles makes a chord of constant length in the other.

Let $k$ = radius of first circle, $r$ = that of the other; then length of chord $= 2\sqrt{(r^2-k^2)}$. In our question $r=0$, therefore chord $= 2k\sqrt{(-1)}$.

2. *Otherwise:* the intercept in question being cut harmonically by every pair of rectangular diameters of the circle, and any pair of the latter intercepting on the tangent two segments measured from its point of contact whose rectangle $= -k^2$; therefore, &c.

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3103. (Proposed by J. J. WALKER, M.A.)—Through B, C, angles of a spherical triangle ABC, draw arcs perpendicular to AB, AC respectively, and meeting in D. Let AE, AF be arcs equally inclined to AB, AC, and meeting, the former the base BC in E, the latter a perpendicular arc through D in F; prove that $\tan AE \,.\, \tan AF = \tan b \,.\, \tan c$,

$$\tan^2 AD = \frac{\tan^2 b + \tan^2 c - 2\tan b \,.\, \tan c \cos A}{\sin^2 A}.$$

*Solution by* MATTHEW COLLINS, B.A.

Project the spherical figure on a plane touching the sphere at A, the centre of the sphere being the centre of projection, and B′C′D′E′F′G′ denoting the projections *in plano* of the points B, C, D, &c.; then the plane angles AB′D′, AC′D′, AF′D′, AG′B′ will obviously be right angles; also

$$\angle BAC = B'AC', \text{ and } B'AE' = BAE \;(\therefore \text{ by hyp.}) = CAF = C'AF'.$$

Hence the circle whose diameter is AD′ passes through B′C′F′, but on the plane figure $AE'.AF' = AB'.AC'$, that is, $\tan AE \,.\, \tan AF = \tan b \tan c$.

Again, on the plane $AD'.\sin A = B'C'$,

and $(B'C')^2 = AB'^2 + AC'^2 - 2AB'.AC'\cos A$, $\therefore = (AD'\sin A)^2$,

that is, $\tan^2 AD \sin^2 A = \tan^2 b + \tan^2 c - 2\tan b \tan c \cos A$.

NOTE.—Since $B'C'.AG' = 2\triangle AB'C' = AB'.AC'\sin A$, therefore

$$AB'^2.AC'^2\sin^2 A = AG'^2.B'C'^2 = AG'^2(AB'^2 + AC'^2 - 2AB'.AC'\cos A),$$

that is, $\tan^2 b \tan^2 c \sin^2 A = \tan^2 p\,(\tan^2 b + \tan^2 c - 2\tan b \tan c \cos A)$,

which, divided by $\tan^2 b \tan^2 c \tan^2 p \sin^2 A$, gives

$$\cot^2 p = \frac{\cot^2 b + \cot^2 c - 2\cot b \cot c \cos A}{\sin^2 A}, \text{ \&c. \&c.}$$

[Mr. WALKER's solution is given on p. 39 of Vol. XV. of the *Reprint.*]

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3495. (Proposed by C. M. INGLEBY, LL.D.)—Find a general expression for any odd number in terms of ascending powers of 2.

*Solution by the* PROPOSER.

1. *Synthetically.*—Since every odd is of the form $2p_1 \pm 1$, where $p_1$ may be odd or even and therefore of the form $2p_2$ or $2p_2 \pm 1$, it follows that every odd is of the form $4p_2 \pm 1$ or $4p_2 + 2 \pm 1$. The same remark applies to $p_2$ as to $p_1$, so that we obtain the form $8p_3 + 4 \pm 2 \pm 1$. Continuing the process indefinitely, we arrive at the general expression

$$2^n p_n + 2^{n-1} \pm 2^{n-2} \pm \ldots \pm 4 \pm 2 \pm 1.$$

Thus, if $n = 3$, the formula is $8p_3 + 4 \pm 2 \pm 1$, as before; and making

$$\begin{array}{lcl} p_3 = 0, & \text{we get} & 1,\ 3,\ 5,\ 7; \\ p_3 = 1, & ,, & 9, 11, 13, 15; \\ p_3 = 2, & ,, & 17, 19, 21, 23; \\ \&c. & & \&c. \end{array}$$

Evidently, giving the second term the double sign, the formula

$$2^n p_n \pm 2^{n-1} \pm 2^{n-2} \pm \ldots \pm 4 \pm 2 \pm 1$$

gives all the odd numbers without repetition if $p_n$ be successively made 1, 3, 5, 7, 11, &c.

2. *Analytically.*—If $2^{n'} p_{n'} \pm 2^{n'-1} \pm 2^{n'-2} \pm \ldots \pm 4 \pm 2 \pm 1$ be a general formula for any odd number, it may be substituted for $p_n$ in $2^n p_n \pm 2^{n-1} \pm 2^{n-2} \pm \ldots \pm 4 \pm 2 \pm 1$. We thus get

$$2^{n+n'} p_n p'_{n'} \pm 2^{n+n'-1} \pm 2^{n+n'-2} \pm \ldots \pm 2^{n+2} \pm 2^{n+1} \pm 2^n \pm 2^{n-1} \pm 2^{n-2} \pm \ldots \\ \ldots \pm 4 \pm 2 \pm 1,$$

which, being of the same form, proves the formula to be true; and it may be easily proved from it that

$$2^n p_n + 2^{n-1} \pm 2^{n-2} \pm \ldots \pm 4 \pm 2 \pm 1$$

is a general expression for any odd number, if $p_n = 1, 2, 3, 4, 5$, &c.

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**3626.** (Proposed by Professor WOLSTENHOLME.)—An inelastic ball is started on a rough table, so that its lowest point begins to move in the same direction as and with greater velocity than its centre. Show that it will move forward with diminishing velocity for a time $\frac{2}{7\mu g}(V + a\Omega)$ and then roll on uniformly, *or* move forward for a time $\frac{V}{\mu g}$, return with increasing velocity for a time $\frac{2a\Omega - 5V}{7\mu g}$, and then roll uniformly, according as $5V >$ or $< 2a\Omega$; where $V, \Omega$ are the initial linear and angular velocities, $a$ is the radius, and $\mu$ the coefficient of friction.

*Solution by* Professor TOWNSEND, M.A., F.R.S.

Denoting, at any time $t$, by $v$ the velocity of the centre of the ball, by $\omega$ its angular velocity of rotation, and by $u$ the velocity of its point of con-

tact with the table; then, since evidently

$$\frac{dv}{dt} = -\mu g, \quad \frac{d\omega}{dt} = -\frac{5}{2}\frac{\mu g}{a}, \quad u = v + a\omega,$$

therefore $v = \mathrm{V} - \mu g t, \quad \omega = \Omega - \frac{5}{2}\frac{\mu g}{a} t, \quad u = \mathrm{V} + a\Omega - \frac{7}{2}\mu g t.$

Hence, if $u=0$ before $v=0$, that is, if $\frac{2}{7}(\mathrm{V}+a\Omega) < \mathrm{V}$ or $2a\Omega < 5\mathrm{V}$, the ball, having previously advanced with diminishing velocity for the time $\frac{2}{7}\frac{\mathrm{V}+a\Omega}{\mu g}$, will from that out continue to roll forwards with the uniform velocity $\mathrm{V} - \frac{2}{7}(\mathrm{V}+a\Omega) = \frac{1}{7}(5\mathrm{V} - 2a\Omega)$; and if, on the contrary, $v=0$ before $u=0$, that is, if $5\mathrm{V} < 2a\Omega$, it will, having previously advanced with diminishing velocity for the time $\frac{\mathrm{V}}{\mu g}$, first return with increasing velocity until $u=0$, that is, for the time $\frac{2a\Omega - 5\mathrm{V}}{7\mu g}$, and then from that out continue to roll backwards with the uniform velocity $\frac{1}{7}(2a\Omega - 5\mathrm{V})$ for the remainder of its motion. In the particular case when $2a\Omega - 5\mathrm{V} = 0$, the velocities of translation and rotation will cease simultaneously, and the ball, having traversed the space $\frac{1}{2}\frac{\mathrm{V}^2}{\mu g}$ in the time $\frac{\mathrm{V}}{\mu g}$, will come to and continue from that out permanently at rest.

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**3579.** (Proposed by R. W. Genese, B.A.) — Find the locus of the centre of the circle TOP, where OT is a fixed and TP a variable tangent to an ellipse.

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*Solution by* R. Tucker, M.A.

Referring the ellipse to the tangent (axis of $x$) and normal at O, its equation is $\quad ax^2 + 2hxy + by^2 + 2gx = 0 \;\ldots\ldots\ldots\ldots (1).$

The equation to the tangent at $\mathrm{P}(x', y')$ is

$$axx' + h(x'y + xy') + byy' + g(x + x') = 0 \;\ldots\ldots\ldots (2);$$

therefore $\quad \mathrm{OT} = -gx' \div (ax' + hy' + g).$

The equations to perpendiculars through the middle points of OT, OP are $\quad 2x(ax' + hy' + g) = -gx' \ldots\ldots (\alpha), \quad 2yy' + 2xx' = x'^2 + y'^2 \ldots\ldots (\beta);$ and there is the condition of P being on (1). From (1) and ($\alpha$) we obtain

$$(ax + g)\,x'^2 = bxy'^2 \;\ldots\ldots\ldots\ldots (\gamma).$$

Substituting from ($\gamma$) $y' = \mu x'$ in ($\beta$), and we have

$$x'(\mu^2 + 1) = 2\mu y + 2x.$$

Finally we get for the locus required the quartic curve

$$4(ab - h^2)\left\{bx^4 - ax^2y^2 - gxy^2\right\} + 4gh(a+b)x^2y + g^2x^2(a+4b) + 4bg(a+b)x^3 + g^3x + 4g^2hxy = bg^2y^2;$$

or taking the equation given in the solution of Question 3099 (*Reprint,*

Vol. XIV., p. 74), we have, if $\alpha, \beta$ are the coordinates of the centre of the circle, $$2\alpha = \frac{b^2(r'^2+c^2)x'}{b^2x'^2+a^2y'^2}, \quad 2\beta = \frac{a^2(r'^2-c^2)y'}{b^2x'^2+a^2y'^2},$$ with the condition $\frac{hx'}{a^2}+\frac{ky'}{b^2}=1$, $(h, k)$ being the point O.

[In the particular case of a tangent OT at the vertex, this gives the cubic curve $$b^2(2a\alpha+a^2+c^2+4\beta^2)+8a\alpha\beta^2=0.]$$

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**3561.** (Proposed by ELIZABETH BLACKWOOD.)—A point is taken at random in a given quadrant of a circle, and a random line is drawn through it. Find the chance of its cutting the arc.

*Solution by* HUGH MCCOLL.

Let AOBC be the quadrant, and P the random point in it. The probability that P falls in the triangle AOB is $\frac{2}{\pi}$. Given that P falls in this triangle, the chance of the random line cutting OA and OB is $\frac{1}{\pi}\log 2$.* Given that P falls in the segment ACB, the chance of the random line cutting OA and OB is zero. The required chance is therefore
$$1-\frac{2}{\pi^2}\log 2 = \cdot 85954 \text{ nearly.}$$

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**3538.** (Proposed by the Rev. W. H. LAVERTY, M.A.)—A is a conic with centre $\alpha$; C is a circle with centre $\gamma$; the asymptotes of C meet A in four points; of the lines joining these points, two will be real; call these $m$ and $n$; draw two normals from $\gamma$ to A; let S and T be the tangents at the extremities of these normals: show that the four intersections MS, NS, MT, NT will lie on a circle concentric with C.

*Solution by the* PROPOSER.

Reciprocate with respect to $\gamma$; A and C remain as before. The asymptotes of C become the circular points at infinity; and from these we must draw four tangents to A. Of the points of intersection of these four tangents, two will be real, and these are what M and N correspond to; therefore M and N are the foci of A. S and T become points at the extremities of normals to A from $\gamma$; therefore we have to show that the four lines MS, NS, MT, NT touch a circle whose centre is $\gamma$, which we know to be true.

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* See Solutions of Quest. 3385, on pp. 83—86 of Vol. XV. of the *Reprint*, and on p. 30 of Vol. XVI.

**3600.** (Proposed by the Rev. W. ROBERTS, M.A.)—If

$$u = \int_0^1 \frac{e^{ax^2}dx}{1+x^2}, \text{ prove that } 4a\left(\frac{du}{da}+u\right)^2 = 4e^a u - \pi.$$

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I. *Solution by* J. W. L. GLAISHER, B.A., F.R.A.S.

$$\frac{du}{da} + u = \int_0^1 e^{ax^2}dx, \text{ therefore}$$

$$\left(\frac{du}{da}+u\right)^2 = \int_0^1\int_0^1 e^{a(x^2+y^2)}dx\,dy = 2\int_0^{\frac{1}{4}\pi}\int_0^{\sec\theta} e^{ar^2} r\,dr\,d\theta$$

$$= \frac{1}{a}\int_0^{\frac{1}{4}\pi}\left(e^{a\sec^2\theta}-1\right)d\theta = \frac{e^a}{a}\int_0^{\frac{1}{4}\pi} e^{a\tan^2\theta}d\theta - \frac{\pi}{4a}$$

$$= \frac{e^a}{a}\int_0^1 \frac{e^{ax^2}}{1+x^2}dx - \frac{\pi}{4a} = \frac{e^a u}{a} - \frac{\pi}{4a}.$$

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II. *Solution by* J. J. WALKER, M.A.

It easily appears that $\frac{du}{da} + u = \int_0^1 e^{ax^2}dx = v$, say.

Integrating by parts, we shall have

$$v = \left[xe^{ax^2}\right]_0^1 - 2a\int_0^1 x^2 e^{ax^2}dx = e^a - 2a\frac{dv}{da},$$

or $\qquad 2a\frac{dv}{da} + v = e^a$; that is, $2a\frac{d^2u}{da^2} + (2a+1)\frac{du}{da} + u = e^a$.

Writing this in the form $\left(2a\frac{d}{da}+1\right)\left(\frac{du}{da}+u\right) = e^a$,

and observing that, $\phi$ being any subject,

$$\frac{d\,.\,a\phi^2}{da} = \phi\left(2a\frac{d}{da}+1\right)\phi, \text{ also that } e^a\left(\frac{du}{da}+u\right) = \frac{d\,.\,e^a u}{da},$$

it is evident that both sides of the equation become perfect differentials after multiplication by $\frac{du}{da}+u$. Accordingly, multiplying and integrating, $a\left(\frac{du}{da}+u\right)^2 = e^a u + c$. To determine the arbitrary constant $c$, when

$a=0$, $\qquad \frac{du}{da} + u = 1$, and $u = \int_0^1 \frac{dx}{1+x^2} = \frac{1}{4}\pi$.

Putting then $a=0$, $e=1$, $u=\frac{1}{4}\pi$ in the preceding equation, $0 = \frac{1}{4}\pi + c$, or $c = -\frac{1}{4}\pi$.

**3273.** (Proposed by A. B. Evans, M.A.)—Find the surface of revolution all points of which are equally illumined by a given luminous straight line.

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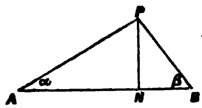
*Solution by the* Rev. J. L. Kitchin, M.A.




Let AB be the given luminous right line. It is clear that the curve, whatever it is, must be symmetrical about AB. We may therefore take AB as axis of $x$, and the extremity A as origin of coordinates. Let $AB=a$, and let P be any point on the curve. Draw the ordinate PN. The illumination from AB on P will be the same as that of the arc of the circle, intercepted between AP, BP, whose centre is P and radius PN;

therefore illumination at $P = \dfrac{C.PN(\pi-\alpha-\beta)}{PN^2}$, and this is constant;

therefore $$\frac{\pi-\alpha-\beta}{y}=c, \quad\text{or}\quad \pi-cy=\alpha+\beta;$$

therefore $$-\tan cy=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\dfrac{y}{x}+\dfrac{y}{a-x}}{1-\dfrac{y^2}{x(a-x)}}=\frac{ay}{x(a-x)-y^2};$$

therefore $y^2-x(a-x)=ay\cot cy$ is the generating curve required.

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**3610.** (Proposed by R. Tucker, M.A.)—If tangents be drawn to $b^2x^2+a^2y^2=a^2b^2$ from any point on $x^2-y^2=a^2-b^2$ or $x^2+y^2=a^2+b^2$, and normals be drawn to the ellipse at the points of contact, these will intersect respectively on the curves

$$\left(\frac{b^2x^2-a^2y^2}{b^2x^2+a^2y^2}\right)^2=\frac{x^2-y^2}{a^2-b^2} \quad\text{or}\quad =(x^2+y^2)\frac{a^2+b^2}{a^4e^4}.$$

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*Generalized Solution by* Professor Wolstenholme.

If $\theta,\phi$ be the excentric angles of two points of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, the point of intersection of the normals is given by the equations

$$X\equiv\frac{ax}{a^2-b^2}=\frac{\cos\theta\cos\phi\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)},\quad Y\equiv\frac{by}{b^2-a^2}=\frac{\sin\theta\sin\phi\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)};$$

and if $\theta,\phi$ be connected by the equation $A\cos\theta\cos\phi+B\sin\theta\sin\phi=C$, the locus of the point of intersection of the normals will be found by eliminating $p, q$ from the three equations

$$X=\frac{p(1-q^2)}{p^2+q^2},\quad Y=\frac{q(1-p^2)}{p^2+q^2}.\quad A(1-q^2)+B(1-p^2)-C(p^2+q^2),$$

where $$p \equiv \frac{\cos \frac{1}{2}(\theta+\phi)}{\cos \frac{1}{2}(\theta-\phi)}, \quad q \equiv \frac{\sin \frac{1}{2}(\theta+\phi)}{\cos \frac{1}{2}(\theta-\phi)}.$$

Now $$X^2(A+C)+Y^2(B+C) =$$

$$\frac{(A+C)p^2+(B+C)q^2-2p^2q^2[A+C+(B+C)]+p^2q^2[(A+C)q^2+(B+C)p^2]}{(p^2+q^2)^2}$$

$$= \frac{[(A+C)p^2+(B+C)q^2][(A+C)q^2+(B+C)p^2]-p^2q^2(A+B+4C)(A+B)}{(p^2+q^2)^2(A+B)}$$

$$= \frac{(A+C)(B+C)(p^4+q^4)+p^2q^2[(A+C)^2+(B+C)^2-(A+B)^2-4C(A+B)]}{(A+B)(p^2+q^2)^2}$$

$$= \frac{(A+C)(B+C)(p^2+q^2)^2-4p^2q^2(BC+CA+AB)}{(A+B)(p^2+q^2)^2};$$

therefore $$\frac{1}{A+B}-\frac{X^2}{B+C}-\frac{Y^2}{A+C} = \frac{4(BC+CA+AB)}{(B+C)(C+A)(A+B)}\frac{p^2q^2}{(p^2+q^2)^2}.$$

In the important case where $BC+CA+AB=0$, this gives at once the locus of the intersection of the two normals. If this relation be not satisfied, we have

$$\frac{4p^2q^2}{(p^2+q^2)^2} = \frac{(B+C)(C+A)(A+B)}{(BC+CA+AB)}\left\{\frac{1}{A+B}-\frac{X^2}{B+C}-\frac{Y^2}{A+C}\right\} \equiv Z^2.$$

Now $$2XY = \frac{2pq}{(p^2+q^2)^2}(1-q^2)(1-p^2) = \frac{Z}{p^2+q^2}(1-p^2-q^2+p^2q^2),$$

$$X^2+Y^2 = \frac{1}{p^2+q^2}-Z^2+\frac{p^2q^2}{p^2+q^2},$$

or $$X^2+Y^2+Z^2-1 = \frac{(1-p^2)(1-q^2)}{p^2+q^2} = \frac{2XY}{Z},$$

or the locus is $$4X^2Y^2 = Z^2(X^2+Y^2+Z^2-1)^2,$$

$Z$ having the meaning given to it above.

If then tangents be drawn to the conic $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ from points on the conic $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}=1$, the locus of the point of intersection of normals at the points of contact is in general $4X^2Y^2 = Z^2(X^2+Y^2+Z^2-1)^2$,

where $$X(a^2-b^2) \equiv ax, \quad Y(b^2-a^2) = by,$$

$$Z^2 = \frac{(B+C)(A+C)-(A+B)(A+C)X^2-(B+C)(A+B)Y^2}{BC+CA+AB},$$

A, B, C denoting $-\dfrac{a^2}{a'^2}+\dfrac{b^2}{b'^2}+1$, $\dfrac{a^2}{a'^2}-\dfrac{b^2}{b'^2}+1$, and $\dfrac{a^2}{a'^2}+\dfrac{b^2}{b'^2}-1$ respectively. If $\dfrac{a}{a'}\pm\dfrac{b}{b'}\pm 1=0$, $BC+CA+AB=0$, and the locus becomes

$$\frac{a'^2X^2}{a^2}+\frac{b'^2Y^2}{b^2} = 1, \quad \text{or} \quad a'^2x^2+b'^2y^2 = (a^2-b^2)^2;$$

this being, of course, the condition that triangles can be circumscribed and inscribed to the two conics, and the locus being that of the intersection of the normals at the points of contact of any such triangle.

If $C=0$, we have $X^2+Y^2+Z^2-1 = -\frac{AX^2}{B} - \frac{BY^2}{A}$,

and the equation of the locus may be written

$$\left(\frac{A^2X^2-B^2Y^2}{A^2X^2+B^2Y^2}\right)^2 = \frac{(AX^2+BY^2)(A+B)}{AB},$$

or
$$\left(\frac{b^2x^2}{b'^4} - \frac{a^2y^2}{a'^4} \div \frac{b^2x^2}{b'^4} + \frac{a^2y^2}{a'^4}\right)^2 = \frac{a'^2x^2+b'^2y^2}{(a^2-b^2)^2}.$$

In the particular case given, $a'^2 = b'^2 = a^2+b^2$, or $a'^2 = -b'^2 = a^2-b^2$; and the corresponding loci are

$$\left(\frac{b^2x^2-a^2y^2}{b^2x^2+a^2y^2}\right)^2 = \frac{(a^2+b^2)(x^2+y^2)}{(a^2-b^2)^2} \quad \text{or} \quad = \frac{x^2-y^2}{a^2-b^2}.$$

If $\frac{a^2}{a'^2} = \frac{b^2}{b'^2} = \frac{1}{2}$, the locus is $\frac{(a^2x^2-b^2y^2)^2}{(a^2x^2+b^2y^2)^2} = \frac{2}{(a^2-b^2)^2}$,

the locus of the intersection of normals at the ends of conjugate diameters.

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II. *Solution by the* Proposer; Rev. W. H. Laverty, M.A.; *and others.*

If $(h, k)$ be the points from which tangents are drawn to the ellipse, and $(x_1, y_1)$, $(x_2, y_2)$ the points of contact, then we know that

$$x_1+x_2 = \frac{2ha^2b^2}{a^2k^2+b^2h^2}, \quad x_1x_2 = \frac{a^4(b^2-k^2)}{a^2k^2+b^2h^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

and a symmetrical result for $y$.

Now the normal at $(x_1, y_1)$ is $y-y_1 = \frac{a^2y_1}{b^2x_1}(x-x_1)$,

which can be put into the form

$$\frac{y}{y_1} - 1 = \frac{a^2}{b^2}\left(\frac{x}{x_1} - 1\right).$$

Similarly for $(x_2, y_2)$, by adding and subtracting, and putting the values obtained from $(\alpha)$, we have

$$\frac{b^2hx}{b^2-k^2} - \frac{a^2ky}{a^2-h^2} = a^2-b^2, \quad \frac{hy}{a^2-h^2} = -\frac{kx}{b^2-k^2} \quad \ldots\ldots\ldots (\beta, \gamma).$$

If $(h, k)$ be a point on $x^2-y^2 = a^2-b^2$, we get from these equations

$$\left(\frac{b^2x^2-a^2y^2}{b^2x^2+a^2y^2}\right) = \frac{x^2-y^2}{a^2-b^2},$$

and if it be a point on $x^2+y^2 = a^2+b^2$ we get similarly

$$\left(\frac{b^2x^2-a^2y^2}{b^2x^2+a^2y^2}\right)^2 = (x^2+y^2)\frac{a^2+b^2}{a^4e^4}.$$

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**3643.** (Proposed by J. W. L. Glaisher, B.A., F.R.A.S.)—Prove that

$$\int_0^\infty e^{\frac{\cos ax - b}{1-2b\cos ax+b^2}} \sin\left\{\frac{\sin ax}{1-2b\cos ax+b^2}\right\}\frac{dx}{x} = \frac{\pi}{2}\left(e^{\frac{1}{1-b}} - 1\right) \quad \ldots(1),$$

$$\int_0^\infty e^{\frac{\cos ax - b}{1-2b\cos ax+b^2}} \cos\left\{\frac{\sin ax}{1-2b\cos ax+b^2}\right\}\frac{dx}{c^2+x^2} = \frac{\pi}{2c}\, e^{\frac{1}{e^{ac}-b}} \quad \ldots (2)$$

$a$ and $c$ being positive, and $b$ numerically $< 1$.

---

*Solution by the* Proposer.

The theorems are particular cases of the following:—

$$\int_0^\infty e^{b_1\cos a_1 x + b_2 \cos a_2 x + \ldots} \sin(b_1 \sin a_1 x + b_2 \sin a_2 x + \ldots)\frac{dx}{x} = \frac{\pi}{2}\left(e^{b_1+b_2+\ldots} - 1\right),$$

$$\int_0^\infty e^{b_1\cos a_1 x + b_2 \cos a_2 x + \ldots} \cos(b_1 \sin a_1 x + b_2 \sin a_2 x + \ldots)\frac{dx}{c^2+x^2}$$

$$= \frac{\pi}{2c}\, e^{b_1 e^{-a_1 c} + b_2 e^{-a_2 c} + \ldots},$$

obtained by means of the known series-summations

$$\frac{\sin ax}{1-2b\cos ax+b^2} = \sin ax + b\sin 2ax + b^2 \sin 3ax + \ldots,$$

$$\frac{\cos ax - b}{1-2b\cos ax+b^2} = \cos ax + b\cos 2ax + b^2 \cos 3ax + \ldots.$$

The truth of the general theorems, which in the above form are due to Biesens De Haan (*Transactions of the Haarlem Society of Science*, tom. XVII., 1862), can be seen as follows:—

Taking the first of them, and writing the sine in its exponential form, the theorem becomes

$$\frac{1}{2i}\int_0^\infty \left(e^{b_1 e^{a_1 x i}} e^{b_2 e^{a_2 x i}} \ldots - e^{-b_1 e^{a_1 x i}} e^{-b_2 e^{a_2 x i}} \ldots\right)\frac{dx}{x} = \frac{\pi}{2}\left(e^{b_1} e^{b_2} \ldots - 1\right).$$

On the left-hand side, the coefficient of $\frac{b_1^{n_1}}{\lfloor n_1} \cdot \frac{b_2^{n_2}}{\lfloor n_2} \ldots$ is

$$\frac{1}{2i}\int_0^\infty \left(e^{(n_1 a_1 + n_2 a_2 + \ldots) xi} - e^{-(n_1 a_1 + n_2 a_2 + \ldots) xi}\right)\frac{dx}{x}$$

$$= \int_0^\infty \frac{\sin(n_1 a_1 + n_2 a_2 + \ldots) x}{x}\, dx = \frac{\pi}{2},$$

($a_1$, $a_2$ ... being positive,) which is clearly the coefficient of the same quantity on the right-hand side.

A similar method establishes the second theorem by use of the integral

$$\int_0^\infty \frac{\cos ax}{c^2+x^2}\, dx = \frac{\pi}{2c}\, e^{-ac}.$$

The form the theorems take if any of the quantities which were required to be positive are negative, is easily seen.

A good many integrals similar to those given above might be obtained, as there are several other theorems involving corresponding series of sines and cosines similar to those noticed; such as, for example,

$$\int_0^\infty e^{b_1 \cos a_1 x + b_2 \cos a_2 x + \dots} \sin(b_1 \sin a_1 x + b_2 \sin a_2 x + \dots + px) \frac{dx}{x} = \tfrac{1}{2}\pi e^{b_1 + b_2 + \dots},$$

$$\int_0^\infty e^{b_1 \cos a_1 x + b_2 \cos a_2 x + \dots} \sin(b_1 \sin a_1 x + b_2 \sin a_2 x + \dots + px) \frac{x\,dx}{c^2 + x^2} = \tfrac{1}{2}\pi e^{b_1 e^{-a_1 c} + b_2 e^{-a_2 c} + \dots - cp},$$

&c. &c.

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**3498.** (Proposed by J. Moffitt.)—If the sides of a triangle be bisected, and from the points of bisection perpendiculars be drawn from the triangle to the circumscribed circle; prove that the sum of the lines so drawn, together with the radius of the inscribed circle, will be equal to the diameter of the circumscribed circle.

*Solution by* Archer Stanley; J. J. Sides; A. B. Evans, M.A.; H. Murphy; *and others.*

1. Let A, B, C be the vertices of an acute-angled triangle; D, E, F the middle points of the sides respectively opposite to these vertices; and G, H, K the feet of the perpendiculars drawn from the latter to the opposite sides. Lastly, O being the centre of the circumscribed circle (radius R), let AL, BM, CN be the perpendiculars let fall respectively on OB, OC, OA. Then, since DOC, BMC, CAK, and BAH are obviously similar right-angled triangles,

$$\left.\begin{array}{l} \dfrac{\mathrm{OD}}{\mathrm{R}} = \dfrac{\mathrm{BM}}{\mathrm{BC}} = \dfrac{\mathrm{AK}}{\mathrm{CA}} = \dfrac{\mathrm{AH}}{\mathrm{AB}} = \dfrac{\mathrm{BM}+\mathrm{AK}+\mathrm{AH}}{\mathrm{BC}+\mathrm{CA}+\mathrm{AB}}, \\ \text{and in like manner} \\ \dfrac{\mathrm{OE}}{\mathrm{R}} = \dfrac{\mathrm{BK}}{\mathrm{BC}} = \dfrac{\mathrm{CN}}{\mathrm{CA}} = \dfrac{\mathrm{BG}}{\mathrm{AB}} = \dfrac{\mathrm{BK}+\mathrm{CN}+\mathrm{BG}}{\mathrm{BC}+\mathrm{CA}+\mathrm{AB}}, \\ \dfrac{\mathrm{OF}}{\mathrm{R}} = \dfrac{\mathrm{CH}}{\mathrm{BC}} = \dfrac{\mathrm{CG}}{\mathrm{CA}} = \dfrac{\mathrm{AL}}{\mathrm{AB}} = \dfrac{\mathrm{CH}+\mathrm{CG}+\mathrm{AL}}{\mathrm{BC}+\mathrm{CA}+\mathrm{AB}}. \end{array}\right\} \quad \dots\dots\dots\dots (1).$$

By addition, we find readily that

$$\frac{\mathrm{OD}+\mathrm{OE}+\mathrm{OF}}{\mathrm{R}} = 1 + \frac{\mathrm{AL}+\mathrm{BM}+\mathrm{CN}}{\mathrm{AB}+\mathrm{BC}+\mathrm{CA}} \quad \dots\dots\dots\dots\dots\dots\dots (2).$$

The last of these ratios, however, is equal to $\frac{r}{\mathrm{R}}$, where $r$ is the radius of the inscribed circle, since $(\mathrm{AL}+\mathrm{BM}+\mathrm{CN})\,\mathrm{R}$ and $(\mathrm{AB}+\mathrm{BC}+\mathrm{CA})\,r$ are each expressions for double the area of the triangle. Hence

$$\mathrm{OD}+\mathrm{OE}+\mathrm{OF} = \mathrm{R}+r \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (3);$$

and deducting these equals from 3R, we have at once the theorem to be proved.

If one of the angles of the triangle, say A, were obtuse, the equations (1) would still hold; but instead of (2) we should form the following:—

$$\frac{-\text{OD}+\text{OE}+\text{OF}}{\text{R}} = 1 + \frac{\text{AL}-\text{BM}+\text{CN}}{\text{AB}+\text{BC}+\text{CA}};$$

and (3) would become $\quad -\text{OD}+\text{OE}+\text{OF} = \text{R}+r.$

The theorem to be proved again results from the deduction of these equals from 3R.

If A were a right angle, OD, BM, AK, and AH would all vanish, and the theorem would result from the addition of the two last of the equations (1).

2. Trigonometrically the theorem may be deduced from the well-known formula, $\quad \sin 2\text{A} + \sin 2\text{B} + \sin 2\text{C} = 4 \sin \text{A} \sin \text{B} \sin \text{C};$

for this is readily transformed into

$$\cos \text{A} + \cos \text{B} + \cos \text{C} - 1 = 2\,\frac{\sin \text{A} \sin \text{B} \sin \text{C}}{\sin \text{A} + \sin \text{B} + \sin \text{C}},$$

and this again into

$$1 - (\sin^2 \tfrac{1}{2}\text{A} + \sin^2 \tfrac{1}{2}\text{B} + \sin^2 \tfrac{1}{2}\text{C}) = \frac{\sin \text{A} \sin \text{B} \sin \text{C}}{\sin \text{A} + \sin \text{B} + \sin \text{C}}.$$

But on multiplying both sides of this equation by

$$2\text{R} = \frac{a}{\sin \text{A}} = \frac{b}{\sin \text{B}} = \frac{c}{\sin \text{C}},$$

and transforming slightly the last term, we have

$$2\text{R} - (\tfrac{1}{2}a \tan \tfrac{1}{2}\text{A} + \tfrac{1}{2}b \tan \tfrac{1}{2}\text{B} + \tfrac{1}{2}c \tan \tfrac{1}{2}\text{C}) = \frac{bc \sin \text{A}}{a+b+c} = r,$$

which, on transposition, becomes the theorem to be proved.

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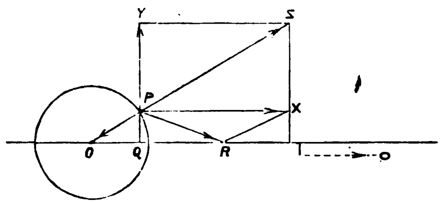
**3594.** (Proposed by Professor TOWNSEND, M.A., F.R.S.)—A mass M of homogeneous fluid, whose elements attract each other and are attracted by a distant body $m$, the force of attraction varying inversely as $n$th power of distance, revolves with a small angular velocity $\omega$ round the axis connecting its centre of gravity O with that of the distant body $o$; if its free surface assume the permanent form of a sphere, show on elementary principles that $\omega^2 = \dfrac{n+1}{d^{n+1}}\, m$, where $d =$ the distance O$o$.

*Solution by the* PROPOSER.




For, if P be any point on the free surface of M, PQ the perpendicular force P on the axis of rotation O$o$, and OR in magnitude and direction $=(n+1)$ OQ; then, since (on well-known elementary principles), if the direct action of M on O is represented by O$o$, the difference of its actions on P and O is represented

in magnitude and direction by PR, or by PX and PO, where PX is parallel and equal to OR (see Fig.); and since the attraction of M on P for the spherical form is of course along PO, for the permanence of the free surface in the spherical form, it is necessary and sufficient that the resultant PZ of PX and of the centrifugal force PY at P should coincide in direction with the radius OP; therefore at once PY : PX = PQ : OQ, or $\omega^2 . \mathrm{PQ} : \frac{n+1}{d^{n+1}} m . \mathrm{OQ} = \mathrm{PQ} : \mathrm{OQ}$, and therefore, &c.

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**2652.** (Proposed by Professor CAYLEY.)—Find the differential equation of the parallel surfaces of an ellipsoid.

I. *Solution by* SAMUEL ROBERTS, M.A.

The process will be more intelligible if we first of all find the differential equation of the plane parallels of an ellipse.

If a point $(x, y)$ is on the plane parallel of $\frac{x^2}{a} + \frac{y^2}{b} - 1 = 0$, its corresponding point on the ellipse, being at a constant distance $r$ on the normal of the parallel, is represented by

$$x - \frac{rp}{(1+p^2)^{\frac{1}{2}}}, \quad y + \frac{r}{(1+p^2)^{\frac{1}{2}}}; \quad \text{where } p = \frac{dy}{dx}.$$

Hence $$\frac{1}{a}\left(x - \frac{rp}{(1+p^2)^{\frac{1}{2}}}\right)^2 + \frac{1}{b}\left(y + \frac{r}{(1+p^2)^{\frac{1}{2}}}\right)^2 - 1 = 0 \quad\text{(1)}.$$

The general solution of (1) represents all the circles of radius $r$ having their centres on the ellipse. The singular solution represents the particular parallel to the modulus $r$.

But let us now impose the further condition that the curves have a common normal at corresponding points. Then we have

$$p = -b\left(x - \frac{rp}{(1+p^2)^{\frac{1}{2}}}\right) \div a\left(y + \frac{r}{(1+p^2)^{\frac{1}{2}}}\right) \quad\text{(2)};$$

eliminating $r$ from (1) and (2), we get $(x+py)(ap^2+b) - p^2(b-a)^2 = 0$, which is the differential equation of all plane parallels of the ellipse.

The equation of a parallel (modulus $r$) is the discriminant of

$$\phi(\theta) = \frac{\theta x^2}{\theta+a} + \frac{\theta y^2}{\theta+b} - \theta - r^2 = 0 \quad\text{(3)},$$

or $$\frac{ax^2}{\theta+a} + \frac{by^2}{\theta+b} + \theta + r^2 - x^2 - y^2 = 0;$$

that is to say, the parallel is represented by (3) and

$$\psi(\theta) = \frac{ax^2}{(\theta+a)^2} + \frac{by^2}{(\theta+b)^2} - 1 = 0 \quad\text{(4)}.$$

Differentiating (3) and (4), we have

$$2\theta\left\{\frac{x}{\theta+a} + \frac{yp}{\theta+b}\right\} + \phi'(\theta)\left(\frac{d\theta}{dx} + \frac{d\theta}{dy}p\right) = 0,$$

$$\frac{d}{dx}\psi(\theta) + \frac{d}{dy}\psi(\theta) . p + \psi'(\theta)\left(\frac{d\theta}{dx} + \frac{d\theta}{dy}p\right) = 0;$$

and since $\phi'(\theta) = 0$, the available condition is $\dfrac{x\,dx}{\theta+a} + \dfrac{y\,dy}{\theta+b} = 0$ ......(5).

Eliminating $\theta$ from (4) and (5), we get the same result as before.

Proceeding in an inverse order with the case of the ellipsoid, we have the equation of a parallel (modulus $r$) represented analogously by

$$\phi(\theta) = \frac{\theta x^2}{\theta+a} + \frac{\theta y^2}{\theta+b} + \frac{\theta z^2}{\theta+c} - \theta - r^2 = 0 \quad \text{....................(6)},$$

$$\psi(\theta) = \frac{ax^2}{(\theta+a)^2} + \frac{by^2}{(\theta+b)^2} + \frac{cz^2}{(\theta+c)^2} - 1 = 0 \quad \text{................(7)}.$$

Differentiating these totally with regard to $x, y, z, \theta$, we get the condition

$$\frac{x\,dx}{\theta+a} + \frac{y\,dy}{\theta+b} + \frac{z\,dz}{\theta+c} = 0 \quad \text{...........................(8)};$$

and eliminating $\theta$ from (7) and (8), we get the differential equation of all the parallels as the resultant of a sextic and a quadratic equation. Further to explain this, we may revert to the process used at the outset.

Writing $p$ for $\dfrac{dz}{dx}$, $q$ for $\dfrac{dz}{dy}$, P for $(p^2+q^2+1)^{\frac{1}{2}}$, we have the point $\left(x - \dfrac{rp}{\mathrm{P}},\ y - \dfrac{rq}{\mathrm{P}},\ z + \dfrac{r}{\mathrm{P}}\right)$ on the ellipsoid corresponding to the point $(x, y, z)$ on the parallel. Hence the condition

$$\frac{1}{a}\left(x - \frac{rp}{\mathrm{P}}\right)^2 + \frac{1}{b}\left(y - \frac{rq}{\mathrm{P}}\right)^2 + \frac{1}{c}\left(z + \frac{r}{\mathrm{P}}\right)^2 - 1 = 0 \quad \text{.........(9)}.$$

The complete solution of this equation represents all spheres of radius $r$ having their centres on the ellipsoid. The general solution represents all the parallels (modulus $r$) of curves traced on the ellipsoid (tubular surfaces). The singular solution represents the particular parallel (modulus $r$).

But now impose the further condition that the ellipsoid and the surfaces have a common normal at corresponding points, so that

$$p = \frac{-c\left(x - \dfrac{rp}{\mathrm{P}}\right)}{a\left(z + \dfrac{r}{\mathrm{P}}\right)}, \qquad q = \frac{-c\left(y - \dfrac{rq}{\mathrm{P}}\right)}{b\left(z + \dfrac{r}{\mathrm{P}}\right)} \quad \text{............ (10, 11)};$$

and we have further the universal equality $dz = p\,dx + q\,dy$ ............(12).

The elimination of $p, q, r$ from (9), (10), (11), and (12) ought to give the previous result as the differential equation. To show this, we may compare (7) with (9). These equations become identical by the equalities

$$x - \frac{rp}{\mathrm{P}} = \frac{ax}{\theta+a},\quad y - \frac{rq}{\mathrm{P}} = \frac{by}{\theta+b},\quad z + \frac{r}{\mathrm{P}} = \frac{cz}{\theta+c}.$$

But substituting the value of $\dfrac{r}{\mathrm{P}}$, thence obtained in terms of $\theta$, in (10) and (11), we get $\quad p = \dfrac{-(\theta+c)\,x}{(\theta+a)\,z},\quad q = \dfrac{-(\theta+c)\,y}{(\theta+b)\,z}$,

and by (12) regain the equation (8).

II. *Solution by* Professor TOWNSEND.

Since, from the equation of the ellipsoid, for any point $(x, y, z)$ on the parallel,

$$\left(\frac{x \mp k \cos\alpha}{a}\right)^2 + \left(\frac{y \mp k \cos\beta}{b}\right)^2 + \left(\frac{z \mp k \cos\gamma}{c}\right)^2 = 1,$$

where $\pm k =$ the parameter of the parallel, and $\alpha, \beta, \gamma$ the direction angles of its normal at $x, y, z$; therefore, substituting in this equation for $\cos\alpha$, $\cos\beta$, $\cos\gamma$ their values $\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}$, divided each by the square root of the sum of their squares, and clearing the result of the radical involved, we have at once the differential equation required; which, as the radical enters only as the divisor of $k$ in the uncleared equation, is consequently the same for the two parallels equidistant in opposite directions by the intervals $\pm k$ from the original surface.

By a similar process, substituting for $x, y, z$, in the equation of any surface, $x \pm k \cos\alpha$, $y \pm k \cos\beta$, $z \pm k \cos\gamma$, changing $\cos\alpha$, &c., into the above values, and clearing the result of the radical involved, we obtain the common differential equation of the two parallels equidistant by the intervals $\pm k$ from the surface.

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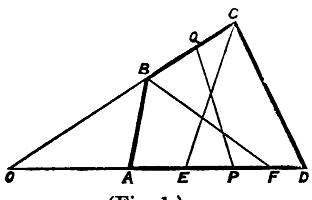
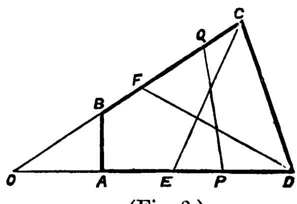
**3571.** (Proposed by the EDITOR.)—ABCD is a quadrilateral, P a point taken at random in AD, and Q a point taken at random in BC; find the probability that the quadrilateral ABQP is less than half of ABCD; and thence show that, if the quadrilateral ABCD becomes the triangle ACD, by the coalescing of A and B, the probability that the triangle AQP is less than half of ACD is $\frac{1}{2}(1 + \log 2)$.

I. *Solution by* G. S. CARR.




(Fig. 1.)




(Fig. 2.)

If AD and BC are parallel, the probability is obviously $\frac{1}{2}$.

If they are not parallel, let them meet, produced, in O. All possible quadrilaterals will be included in the two following cases:—

*First.*—The lines drawn from B and C to bisect the quadrilateral may cut the opposite side. Thus let CE and BF (fig. 1) be bisecting lines.

Let $a, b, c, d, e, x, y$ be the distances of the points A, B, C, D, E, P, Q from O; and let $p$ be the probability required.

If ABQP be less than $\frac{1}{2}$ ABCD, we must have

$$xy < \tfrac{1}{2}(ab+cd);$$

and therefore $\quad y < \dfrac{ab+cd}{2x}, \quad e = \dfrac{ab+cd}{2c}, \quad f = \dfrac{ab+cd}{2b};$

therefore $\quad p = \displaystyle\int_a^e\int_b^c \frac{dx\,dy}{(d-a)(c-b)} + \int_e^f\int_b^u \frac{dx\,dy}{(d-a)(c-b)},$

in which $u = \dfrac{ab+cd}{2x}$ the limit of $y$ when $x$ lies between $e$ and $f$.

Integrating, we find $\quad p = \dfrac{\frac{1}{2}(ab+cd)(\log c - \log b) - a(c-b)}{(d-a)(c-b)}.$

The probability of PQCD being less than $\frac{1}{2}$ ABCD is of course $1-p$.

*Secondly.*—The bisecting lines drawn from C and D, as CE, DF (fig. 2), may cut adjacent sides. In this case $1-p$ is more easily calculated than $p$, the former requiring one integral, and the latter two.

Here we have $\quad 1-p = \displaystyle\int_e^d\int_u^c \frac{dx\,dy}{(d-a)(c-b)},$

from which $\quad p = 1 - \dfrac{(cd-ab)-(ab+cd)\left\{\log 2cd - \log(ab+cd)\right\}}{2(d-a)(c-b)}.$

In this last case, if A and B move up to O, so that the quadrilateral ABCD becomes the triangle ACD, we have $a=b=0$, and

$$p = \tfrac{1}{2}(1+\log 2) = \cdot 84657309, \quad 1-p = \tfrac{1}{2}(1-\log 2) = \cdot 15342691$$

II. *Solution by* Stephen Watson.

Produce DA, CB to meet in O, and bisect the quadrilateral by the lines CE, DF [see fig. 2 in the foregoing solution]. Put OA $= a$, OB $= b$, OC $= c$, OD $= d$, OP $= x$, OQ $= y$. Then, when ABQP $< \frac{1}{2}$ ABCD,

$$xy - ab < cd - xy, \text{ or } xy < \tfrac{1}{2}(ab+cd) = m \text{ (suppose)},$$

therefore $\quad \text{OE} = \dfrac{m}{c}, \quad \text{OF} = \dfrac{m}{d}.$

When P lies on AE, Q may lie anywhere on BC, and the number of positions P and Q can take is

$$\left(\frac{m}{c} - a\right)(c-b) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

When P lies on ED, the number of positions is

$$\int_{\frac{m}{c}}^{d} \left(\frac{m}{x} - b\right) dx = m \log\frac{cd}{m} - b\left(d - \frac{m}{c}\right) \ldots\ldots\ldots\ldots (2).$$

Hence the required chance is

$$p = \frac{(1)+(2)}{(d-a)(c-b)} = \frac{1}{(d-a)(c-b)}\left\{m\log\frac{cd}{m} + m - bd - a(c-b)\right\} \ldots \;(3),$$

[which, in a slightly different form, agrees with Mr. Carr's result].

When the quadrilateral becomes a triangle, $a = b = 0$, and the chance becomes $\quad \frac{1}{2}(1+\log 2)$.

## On Hutton's Rule for Approximating to the Roots of Numbers.
*By* C. W. Merrifield, F.R.S.

This rule, which I have used in my *Technical Arithmetic* as a substitute for the extraction of the Cube Root, is not to be found in our modern Algebras, and I have therefore thought that it might be convenient to give some account of it. It appears to have been in familiar use at the end of the last and the beginning of the present century. So far as I can find, it was first given by Hutton in his *Mathematical Tracts*. It is repeated as a well-known formula in Vega's and Barlow's *Tables*, in Hind's *Algebra*, in Butler's *Course of Mathematics*, and in several other works of the same period. Butler gives it considerable prominence, remarking that it is "perhaps more convenient for memory and operation than any other rule that has been discovered."

The rule is as follows:—If $a$ be an approximate $n$th root of N, then $\dfrac{(n+1)\,\mathrm{N}+(n-1)\,a^n}{(n-1)\,\mathrm{N}+(n+1)\,a^n}\,.\,a$ is a nearer approximation. It is easily established by the binomial theorem. If we write $\mathrm{N}=(a+x)^n=a^n+na^{n-1}x$, and stop at the second term, neglecting all that follow, we have

$$a+x_1=\frac{\mathrm{N}+(n-1)\,a^n}{na^n}\,.\,a,$$

which is Newton's approximation.

If we now proceed to the third term of the expansion, and write it in the form $\mathrm{N}=a^n+x_2\left\{na^{n-1}+\tfrac{1}{2}n\,(n-1)\,a^{n-2}x_1\right\}$, and give $x_1$ the value previously obtained, namely $\dfrac{\mathrm{N}-a^n}{na^{n-1}}$, we easily obtain

$$a+x_2=\frac{(n+1)\,\mathrm{N}+(n-1)\,a^n}{(n-1)\,\mathrm{N}+(n+1)\,a^n}\,.\,a,$$

which is Hutton's rule. When $n$ is odd, 2 divides out; and in the particular case of the cube root, we get $a+x_2=\dfrac{2\mathrm{N}+a^3}{\mathrm{N}+2a^3}\,.\,a.$

There is nothing to prevent the process being carried on to the third and fourth terms; but there is greater advantage in the repeated use of this formula than in the direct use of more complicated expressions.

In 1862 (see the *Philosophical Transactions* for that year, p. 430) I extended the process to the general determination of $x$ from the equation

$$\mathrm{N}=a_0+a_1x+a_2x^2+a_3x^3+\dots\ \dots\dots\dots\dots\dots \quad (1),$$

by successive substitution, as follows:—Let us write

$$\mu=\mathrm{N}-a_0=a_1x_1=a_1x_2+a_2x_2x_1=a_1x_3+a_2x_3x_2+a_3x_3x_2x_1\ \dots\dots \quad (2),$$

and so forth; then, if we make

$$(\mu-a_1x-a_2x^2-a_3x^3-\dots)^{-1}=\lambda_0+\lambda_1x+\lambda_2x^2+\lambda_3x^3\dots\ \dots\dots\dots \quad (3),$$

and determine the coefficients $\lambda_0$, $\lambda_1$, $\lambda_2$, &c. by actual division or by any equivalent method, I say that we shall also have

$$\mu=\frac{1}{\lambda_0},\quad x_1=\frac{\lambda_0}{\lambda_1},\quad x_2=\frac{\lambda_1}{\lambda_2},\quad x_3=\frac{\lambda_2}{\lambda_3},\ \dots\dots;$$

for if we substitute those values in equation (2), we obtain

$$\mu=\frac{a_1\lambda_0}{\lambda_1}=\frac{a_1\lambda_1+a_2\lambda_0}{\lambda_2}=\frac{a_1\lambda_2+a_2\lambda_1+a_3\lambda_0}{\lambda_3}=\text{\&c.},$$

which are the same equations as we obtain by equating coefficients in (3).

This shows very distinctly the connection between the quantities $x_1, x_2, x_3$, &c., and the coefficients arising out of common division.

In the case of the binomial expansion $N = (a+x)^n$, the approximation is a true one; that is to say, the quantities $x_1, x_2, x_3$, &c., tend to the same limit, at least for the first few terms, which are all that we need. But in many other cases we meet with divergence, and the second approximation is not so good as the first or Newtonian. There is therefore no advantage in its indiscriminate use. The first three terms in the general case are

$$x_1 = \frac{\mu}{a_1}, \quad x_2 = \frac{a_1\mu}{a_2\mu + a_1^2}, \quad x_3 = \frac{a_2\mu^2 + a_1^2\mu}{a_3\mu^2 + 2a_2a_1\mu + a_1^3}.$$

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**3629.** (Proposed by G. M. Minchin, M.A.)—If a particle placed in the focus of a parabola be attracted or repelled by the parabola with a force varying as $r^{-\frac{3}{2}}$, it will remain at rest. The same is the case if the particle be placed at the cusp of a cardioid, and attracted or repelled by a force varying as $r^{-\frac{1}{2}}$. Prove this, and deduce a general theorem of which they are immediate particular cases.

*Solution by* Professor Townsend, M.A., F.R.S.

Any curve included in the equation $r^n = a^n \cos n\theta$, if of uniform density throughout its entire length, will, as is evident from the differential relation $r^{n-1}\,ds = a^n\,d\theta$ which results immediately from that equation, exert no attraction or repulsion upon a particle at the polar origin for the law of force $r^{n-1}$. But for the parabola $n = -\frac{1}{2}$, and for its inverse the cardioid $n = +\frac{1}{2}$, and therefore, &c.

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Note on the Rectangular Hyperbola. *By* C. Taylor, M.A.

To prove that, if a rectangular hyperbola circumscribes a triangle, it passes through the orthocentre.

Let ABC be the triangle, and O its orthocentre. The angle between two diameters is equal to that between their conjugates. Hence it may be shown that the centre C′ is on the nine-point circle. Bisect AO, BC in $a$, $\alpha$; then, $a\alpha$ being a diameter of the circle, C$a$ is at right angles to C$\alpha$, and therefore bisects chords perpendicular to BC. But it bisects AO; therefore O is on the curve.

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**3537.** (Proposed by the Rev. Dr. Booth, F.R.S.)—If $(u, t)$ be the coordinates of the point Q, from which two lines $(r_1, r_2)$ are drawn to the foci of an ellipse, and two tangents on which are let fall the perpendiculars $(p_1, p_2)$ from the centre; prove that $p_1p_2r_1r_2 = a^2u^2 + b^2t^2$.

I. *Solution by* J. J. WALKER, M.A.

The equation to the two tangents is (Salmon's *Conics*, 5th ed., p. 327)

$$(b^2-u^2)x^2+(a^2-t^2)y^2+2tuxy-2b^2tx-2a^2uy+a^2u^2+b^2t^2=0.$$

In the *Cambridge and Dublin Mathematical Journal*, Vol. IX., p. 166, I gave the formula for the product of perpendiculars let fall from $(x', y')$ on the lines whose equation (referred to axes inclined at an angle $\phi$) is

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0,$$

viz.
$$\frac{(Ax'^2+Bx'y'+Cy'^2+Dx'+Ey'+F)\sin^2\phi}{\left\{(A-C)^2+2(A+C)B\cos\phi+B^2+4AC\cos^2\phi\right\}^{\frac{1}{2}}}.$$

In this case, therefore, we have

$$p_1p_2=\frac{a^2u^2+b^2t^2}{\left\{(u^2-t^2+a^2-b^2)^2+4u^2t^2\right\}^{\frac{1}{2}}}=\frac{a^2u^2+b^2t^2}{r_1r_2};\ \text{therefore, \&c.}$$

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II. *Solution by* STEPHEN WATSON.

Let
$$y-u=m(x-t)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
be the equation of either tangent from Q, and eliminating $y$ between (1) and $a^2y^2+b^2x^2=a^2b^2$, in order that the resulting quadratic in $x$ may be a complete square, we must have,

$$(u-mt)^2=a^2m^2+b^2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Also, if $p$ be the perpendicular from the centre upon (1),

$$p^2=\frac{(mt-u)^2}{1+m^2}=\frac{a^2m^2+b^2}{1+m^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Eliminate $m$ from (2) and (3), the result, being a quadratic in $p^2$, is, so far as the first and absolute terms are concerned,

$$\left\{(u^2+t^2+c^2)^2-4c^2t^2\right\}p^4+\ \ldots\ldots\ +(a^2u^2+b^2t^2)^2=0,$$

and the two values of $p^2$ are $p_1^2,\ p_2^2$,

therefore
$$p_1^2p_2^2=\frac{(a^2u^2+b^2t^2)^2}{(u^2+t^2+c^2)^2-4c^2t^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Also $\quad r_1^2r_2^2=\left\{u^2+(c-t)^2\right\}\left\{u^2+(c+t)^2\right\}=(u^2+t^2+c^2)^2-4c^2t^2;$

therefore
$$r_1r_2p_1p_2=a^2u^2+b^2t^2.$$

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**3599.** (Proposed by J. W. L. GLAISHER, B.A.)—Prove that

$$\left(8\int_{p^2}^{\infty}p\,dp\right)^{2i}e^{-2pq}\cos 2pq=q\left(-\frac{d}{q\,dq}\right)^{2i}\frac{e^{-2pq}\cos 2pq}{q},$$

$$\left(8\int_{p^2}^{\infty}p\,dp\right)^{2i}e^{-2pq}\sin 2pq=q\left(-\frac{d}{q\,dq}\right)^{2i}\frac{e^{-2pq}\sin 2pq}{q}.$$

*Solution by the* PROPOSER.

From the known definite integrals

$$\int_0^\infty e^{-\frac{\alpha}{x^2}} \sin\left(2\beta x^2 + \tfrac{1}{4}\pi\right) dx = \pi^{\frac{1}{2}} \frac{e^{-2(\alpha\beta)^{\frac{1}{2}}} \cos 2(\alpha\beta)^{\frac{1}{2}}}{2(2\beta)^{\frac{1}{2}}},$$

$$\int_0^\infty e^{-\frac{\alpha}{x^2}} \sin\left(2\beta x^2 - \tfrac{1}{4}\pi\right) dx = \pi^{\frac{1}{2}} \frac{e^{-2(\alpha\beta)^{\frac{1}{2}}} \sin 2(\alpha\beta)^{\frac{1}{2}}}{2(2\beta)^{\frac{1}{2}}},$$

we obtain

$$\left(-\frac{d}{2d\beta}\right)^{2i} \frac{e^{-2(\alpha\beta)^{\frac{1}{2}}} \cos 2(\alpha\beta)^{\frac{1}{2}}}{\beta^{\frac{1}{2}}} = \left(\int_\alpha^\infty d\alpha\right)^{2i} \frac{e^{-2(\alpha\beta)^{\frac{1}{2}}} \cos 2(\alpha\beta)^{\frac{1}{2}}}{\beta^{\frac{1}{2}}},$$

$$\left(-\frac{d}{2d\beta}\right)^{2i} \frac{e^{-2(\alpha\beta)^{\frac{1}{2}}} \sin 2(\alpha\beta)^{\frac{1}{2}}}{\beta^{\frac{1}{2}}} = \left(\int_\alpha^\infty d\alpha\right)^{2i} \frac{e^{-2(\alpha\beta)^{\frac{1}{2}}} \sin 2(\alpha\beta)^{\frac{1}{2}}}{\beta^{\frac{1}{2}}},$$

and these give the theorems in the question on putting $\alpha = p^2$, $\beta = q^2$.

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**3534.** (Proposed by J. COLLINS.)—A quadrilateral has the sum of its opposite angles equal, and likewise the sum of its opposite sides. If from the middle point of either diagonal perpendiculars are drawn to the sides, the sum of the opposite perpendiculars will also be equal.

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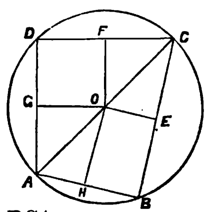
*Solution by* A. B. EVANS, M.A.; T. MITCHESON, B.A.; *and others.*




Let O be the middle point of the diagonal AC of the quadrilateral ABCD, and OH, OE, OF, OG the four perpendiculars upon AB, BC, CD, DA. Then

$$\frac{\sin ACB}{AB} = \frac{\sin ABC}{AC} = \frac{\sin ADC}{AC} = \frac{\sin DAC}{DC} \ldots (1).$$

Multiplying (1) by BC − AB = DC − AD, observing that BC sin ACB = AB sin CAB, and AD sin DAC = CD sin DCA, we obtain

$$\sin CAB - \sin ACB = \sin DAC - \sin DCA;$$

therefore $AO \cdot \sin CAB + CO \cdot \sin DCA = CO \cdot \sin ACB + AO \cdot \sin DAC,$

or $$OH + OF = OE + OG.$$

The same method of proof applies to the four perpendiculars from the middle point of the diagonal BD.

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**3605.** (Proposed by G. M. MINCHIN, M.A.)—Prove that the equation of the polar reciprocal of the first positive central pedal of the quadric

$\frac{x}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, with regard to the quadric, is obtained from the equation of the first negative central pedal of $a^2x^2+b^2y^2+c^2z^2=1$, by the substitution of $\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}$ for $x, y, z$ respectively.

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*Solution by the* PROPOSER.

The first positive pedal of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, is

$$(x^2+y^2+z^2)^2 = a^2x^2+b^2y^2+c^2z^2,$$

he tangent plane to which at $(x_1, y_1, z_1)$ is

$$xx_1(2t-a^2)+yy_1(2t-b^2)+zz_1(2t-c^2)-t^2=0,$$

where $t = x_1^2+y_1^2+z_1^2$. Let the pole of this with regard to the quadric be $(\xi, \eta, \zeta)$; therefore $\frac{x\xi}{a^2}+\frac{y\eta}{b^2}+\frac{z\zeta}{c^2}=1.$

Identifying these equations, we get

$$\frac{t^2\xi}{a^2} = (2t-a^2)\,x_1,\quad \frac{t^2\eta}{b^2} = (2t-b^2)\,y_1,\quad \frac{t^2\zeta}{c^2} = (2t-c^2)\,z_1 \;\ldots\ldots\ldots\; \text{(A)}.$$

Multiplying both sides of these equations (A) by $x_1, y_1, z_1$ respectively, and adding, we obtain $\frac{x_1\xi}{a^2}+\frac{y_1\eta}{b^2}+\frac{z_1\zeta}{c^2}=1$ .............................. (B).

But the system (A) gives also $\frac{x_1\xi}{a^2}=\frac{t^2\xi^2}{a^4(2t-a^2)}$, &c.; therefore (B) becomes

$$\frac{t^2\xi^2}{a^4(2t-a^2)}+\frac{t^2\eta^2}{b^4(2t-b^2)}+\frac{t^2\zeta^2}{c^4(2t-c^2)}=1 \;\ldots\ldots\ldots\ldots\; \text{(C)}.$$

Also, by squaring and adding, the system (A) gives

$$\frac{t^3\xi^2}{a^4(2t-a^2)^2}+\frac{t^3\eta^2}{b^4(2t-b^2)^2}+\frac{t^3\zeta^2}{c^4(2t-c^2)^2}=1 \;\ldots\ldots\ldots\; \text{(D)}.$$

Now (D) is the differential of (C) with regard to the variable $t$; hence the locus of $(\xi, \eta, \zeta)$ is the discriminant of (C).

But (see SALMON's *Geometry of Three Dimensions*, p. 409) the first negative pedal of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is the discriminant of

$$\frac{x^2}{t\left(2-\frac{t}{a^2}\right)}+\frac{y^2}{t\left(2-\frac{t}{b^2}\right)}+\frac{z^2}{t\left(2-\frac{t}{c^2}\right)}=1;$$

and if in (C) we write X, Y, Z for $\frac{\xi}{a^2}, \frac{\eta}{b^2}, \frac{\zeta}{c^2}$, and $\theta$ for $\frac{1}{t}$, (C) becomes

$$\frac{X^2}{\theta(2-\theta a^2)}+\frac{Y^2}{\theta(2-\theta b^2)}+\frac{Z^2}{\theta(2-\theta c^2)}=1,$$

the discriminant of which is the first negative pedal of

$$a^2X^2+b^2Y^2+c^2Z^2=1;\ \text{therefore, \&c.}$$

**3597.** (Proposed by J. J. WALKER, M.A.)—Prove the following extension of Dr. HIRST's Question 3567 :—

The envelope of the skew quadric surface, which passes through a fixed point (L) and through the four lines in which two fixed planes $(m, n)$ are intersected by two other planes passing through a variable line lying in a third fixed plane $(l)$, and each through one of two fixed lines coplanar with one another and with the point (L), is a quadric cone touching the planes $(m, n)$ along the lines in which they are met by the plane $(l)$.

*Solution by the* PROPOSER.

The two fixed lines may be supposed to be determined by the intersection of the two planes $k$ and $k+l$ with a fixed plane $p$, passing through the point L. Then $\lambda p + k$ and $\lambda p + k + l$ will represent two planes passing through those lines respectively and through a variable line in $l$, if $\lambda$ be a variable parameter; and

$$m_1 n_1 (\lambda p + k)(\lambda p + k + l) - k_1 (k_1 + l_1) mn$$

will represent the skew quadric surface passing through L and through the intersections of $\lambda p + k$ and $\lambda p + k + l$ with $m, n$, if $k, l, m, n$ are the values of $k, l, m, n$ respectively when the coordinates of L are substituted in them. The discriminant with respect to $\lambda$ of the quadric above is, after reduction and rejection of the factor $p^2$, which evidently forms no part of the envelope,

$$m_1 n_1 l^2 + 4 (k_1 + l_1) k_1 mn,$$

a quadric cone touching $m, n$ along the lines in which they are met by $l$.

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**3675.** (Proposed by Sir JAMES COCKLE, F.R.S.)—Given; that $a$ and $\beta$ are functions of $x$ only, and that

$$\gamma = f(a, \beta) = \frac{a}{2}\left(\frac{d^2 a}{dx^2} + \frac{d\beta}{dx}\right) - \frac{1}{4}\left(\frac{da}{dx} + \beta\right)\left(\frac{da}{dx} - \beta\right) \quad \text{.........} (1);$$

given also (*Reprint*, Vol. XV. pp. 77—80) the solution of the biordinal

$$a^2 r - t = \beta q + \gamma z \quad \text{.................................} (2).$$

Required the conjugate equations and their respective first integrals.

*Solution by the* PROPOSER.

1. The conjugate equations may be written thus:

$$a^2 r - t = f(a, \ \beta) z + \beta q, \quad (-a)^2 r - t = f(-a, \ \beta) z + \beta q \ \text{......} \ (3, 4),$$

$$a^2 r - t = f(a, -\beta) z - \beta q, \quad (-a)^2 r - t = f(-a, -\beta) z - \beta q \ \text{......} \ (5, 6).$$

Put $\quad f(a, \beta) = \gamma_1 = f(-a, -\beta), \ f(-a, \beta) = \gamma_2 = f(a, -\beta)$ ...... (7, 8);
then (3) and (6) respectively become $\ (\pm a)^2 r - t = \gamma_1 z \pm \beta q$ .........(9, 10);
so (3) and (4) become $\quad (\mp a)^2 r - t = \gamma_2 z \pm \beta q$ ........................(11, 12).
But (compare *Reprint*, Vol. XV., p. 78) we may write the first integral of

(9) in the form $\quad q - ap + \frac{1}{2}\left(\frac{da}{dx} + \beta\right) z + \sqrt{a}\, e^{-\int \frac{\beta dx}{2a}} \phi\left(y - \int \frac{dx}{a}\right)$......(13);

and the first integral of (10) is obtained from (13) by changing the signs of $\alpha$ and $\beta$ therein. The first integral of (11) is obtained by changing the sign of $\alpha$ alone, and that of (12) by changing the sign of $\beta$ alone.

2. When $\beta$ is constant, then $\gamma_1=\gamma_2$, and (9) and (11) are but different modes of writing the same equation. The same thing is true of (10) and (12). Of the equation represented by (9) and (11), or by (10) and (12), we can consequently obtain conjugate first integrals, and thence a second integral containing two arbitrary functions. If $\dfrac{d\alpha}{dx}+\beta=0$ ......... (14),

then $$\gamma_1=0,\quad \gamma_2=\alpha\frac{d^2\alpha}{dx^2},\ \text{and}\int\frac{\beta dx}{2\alpha}=-\log\sqrt{\alpha}\ \ldots\ldots(15,16,17);$$

for it is unnecessary to insert an arbitrary constant, since $\phi$ is arbitrary. The systems (9, 10) and (11, 12) now respectively become

$$(\pm\alpha)^2r-t=\mp\frac{d\alpha}{dx}q\ \ldots\ldots(9,10)_2,\quad (\mp\alpha)^2r-t=\alpha\frac{d^2\alpha}{dx^2}\mp\frac{d\alpha}{dx}q\ \ldots\ldots(11,12)_2.$$

The solution of $(9,10)_2$ is $\quad q\mp\alpha p+\alpha\phi\left(y\mp\int\dfrac{dx}{\alpha}\right)$ ........................ (A),

while that of $(11,12)_2$ is $\quad q\pm\alpha p\mp\dfrac{d\alpha}{dx}z+\phi\left(y\pm\int\dfrac{dx}{\alpha}\right)$ ..................(B);

it being remembered that $\sqrt{(-\alpha)}=\sqrt{(-1)}\sqrt{\alpha}$, and that the $\sqrt{(-1)}$ may be merged in $\phi$. If, in place of (14), we assume $\dfrac{d\alpha}{dx}-\beta=0$ ... (18), we have the same results presented in a different order. Thus $\gamma_1$ and $\gamma_2$ interchange their values as given in (15, 16); and $(12,11)_2$ appear in place of $(9,10)_2$, and $(10,9)_2$ in place of $(11,12)_2$.

3. By treating $p$ and $q$ successively as constant, we may in some cases (see *Reprint*, Vol. XIV., pp. 66, 67) obtain distinct conditions of solubility. In the case of (2), we are led to one and the same condition. Let us, for the sake of illustration, deal with form (4); then, making $p$ constant, we import from Monge's systems the following equations, $dp$ being made zero:

$$dq=-(\beta q+\gamma z)\,dy,\quad dx=-\alpha dy\ \ldots\ldots(\mathrm{C},\mathrm{D}).$$

Let $v$ be some function of $x$ only. Then we have

$$vdq+q\frac{dv}{dx}dx=d(vq)=-\left(\beta v+\alpha\frac{dv}{dx}\right)q\,dy+\frac{\gamma zv}{\alpha}dx\ \ldots\ldots(\mathrm{E}).$$

But $q\,dy=dz-p\,dx$; hence we have

$$d(vq)-p\left(\beta v+\alpha\frac{dv}{dx}\right)dx=\frac{\gamma zv}{\alpha}dx-\left(\beta v+\alpha\frac{dv}{dx}\right)dz\ \ldots\ldots(\mathrm{F}).$$

The sinister of (F) will be a perfect differential provided that

$$\frac{\gamma v}{\alpha}=-\frac{d}{dx}\left(\beta v+\alpha\frac{dv}{dx}\right)\ \ldots\ldots(\mathrm{G}).$$

Suppose that (G) is satisfied, and integrate (F). We have, $p$ being taken

as constant, $$vq-p\int\left(\beta v+\alpha\frac{dv}{dx}\right)dx+\left(\beta v+\alpha\frac{dv}{dx}\right)z=c\ \ldots\ldots(\mathrm{H}),$$

where $c$ is an arbitrary constant. Denote (H) by $u=c$; then we have to

satisfy $$\frac{du}{dq}=\frac{1}{\alpha}\frac{du}{dp},\ \text{or}\ v=-\frac{1}{\alpha}\int\left(\beta v+\alpha\frac{dv}{dx}\right)dx\ \ldots\ldots(\mathrm{I}).$$

Hence we find $$\frac{d}{dx}(\alpha v)+\left(\beta v+\alpha\frac{dv}{dx}\right)=\left(\beta+\frac{d\alpha}{dx}\right)v+2\alpha\frac{dv}{dx}=0\ \ldots\ldots(\mathrm{I}_2).$$

Consequently $$v = \frac{1}{\sqrt{a}}\, e^{-\int\frac{\beta dx}{2a}} \text{ .................................... (J).}$$

Moreover, by (G) and ($I_2$),

$$\gamma = -\frac{a}{v}\frac{d}{dx}\left(\beta v + a\frac{dv}{dx}\right) = \frac{a}{v}\,\frac{d^2(av)}{dx^2} \text{ ........................ (K).}$$

$$= \frac{a}{2}\left(\frac{d^2a}{dx^2} - \frac{d\beta}{dx}\right) - \frac{1}{4}\left(\frac{da}{dx}+\beta\right)\left(\frac{da}{dx}-\beta\right) \text{ ...............(L)}$$

and this is the correct result. For we started from form (4), and regarded $a$ as negative; and (L) is what (1) becomes when we therein replace $a$ by $-a$. The $\gamma$ of (L) is indeed $\gamma_2$.

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**3608.** (Proposed by G. O'Hanlon.)—A triangle of constant area is inscribed in a given ellipse; find the locus of its centre of gravity.

*Solution by the* Rev. W. H. Laverty, M.A.

Properties concerning centre of gravity are of course orthogonally projective.

Now in the circle, if the area of the triangle is that of the maximum triangle, the centre of gravity will always be at the centre. But in other cases, there will be an infinite number of triangles with bases parallel to BC, and areas equal to ABC (all contained between certain limits). And as the system of triangles is made to revolve in the circle, these centres of gravity will between them generate a circular band concentric with the given circle.

Hence, in the ellipse, the locus of the centre of gravity is an elliptic band whose interior and exterior boundaries are ellipses similar, similarly situated, and concentric with the given ellipse.

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**3677.** (Proposed by Professor Cayley.)—Find at any point of a plane curve the angle between the normal and the line drawn from the point to the centre of the chord parallel and indefinitely near to the tangent at the point; and examine whether a like question applies to a point on a surface and the indicatrix section at such point.

I. *Solution by* Professor Townsend, M.A., F.R.S.

In the case of a plane curve whose equation is $u=0$, the angle $\theta$ required being evidently that between the perpendicular $(\alpha\beta)$ to the line $p\xi+q\eta=0$ where $p=\frac{du}{dx}$, $q=\frac{du}{dy}$, and the diameter $(\lambda\mu)$ conjugate to the same line

with respect to the conic $a\xi^2 + 2h\xi\eta + b\eta^2 = 0$, where $a = \frac{d^2u}{dx^2}$, $h = \frac{d^2u}{dx\,dy}$, $b = \frac{d^2u}{dy^2}$; hence since, from the theory of conics,

$$a \cos\lambda + h \cos\mu = k \cos\alpha, \quad h \cos\lambda + b \cos\mu = k \cos\beta,$$

and since consequently

$$\begin{aligned}(ab-h^2)(\cos\alpha\cos\lambda + \cos\beta\cos\mu) &= (ab-h^2)\cos\theta \\ &= k\,(b\cos^2\alpha + a\cos^2\beta - 2h\cos\alpha\cos\beta),\end{aligned}$$

where $\cos\alpha = \frac{p}{(p^2+q^2)^{\frac{1}{2}}}$, $\cos\beta \frac{q}{(p^2+q^2)^{\frac{1}{2}}}$,

$$\frac{ab-h^2}{k} = \left\{(b^2+h^2)\cos^2\alpha + (a^2+h^2)\cos^2\beta - 2\,(a+b)\,h\,.\cos\alpha\cos\beta\right\}^{\frac{1}{2}},$$

therefore $\cos\theta = \dfrac{bp^2 + aq^2 - 2hpq}{\left\{p^2+q^2\right\}^{\frac{1}{2}}\left\{(b^2+h^2)\,p^2 + (a^2+h^2)\,q^2 - 2\,(a+b)\,h\,.pq\right\}^{\frac{1}{2}}}$.

Analogously, in the case of a surface whose equation is $u=0$, the corresponding angle $\theta$ being evidently that between the perpendicular $(\alpha\beta\gamma)$ to the plane $p\xi + q\eta + r\zeta = 0$, where $p = \frac{du}{dx}$, $q = \frac{du}{dy}$, $r = \frac{du}{dz}$, and the diameter $(\lambda\mu\nu)$ conjugate to the same plane with respect to the quadric $a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = 0$, where $a = \frac{d^2u}{dx^2}$, &c.; hence since, from the theory of quadrics,

$$a\cos\lambda + h\cos\mu + g\cos\nu = k\cos\alpha, \quad h\cos\lambda + b\cos\mu + f\cos\nu = k\cos\beta,$$
$$g\cos\lambda + f\cos\mu + c\cos\nu = k\cos\gamma,$$

and since consequently

$$\Delta\,(\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu) = \Delta\,.\cos\theta$$
$$= k\,(\mathrm{A}\cos^2\alpha + \mathrm{B}\cos^2\beta + \mathrm{C}\cos^2\gamma + 2\mathrm{F}\cos\beta\cos\gamma + 2\mathrm{G}\cos\gamma\cos\alpha + 2\mathrm{H}\cos\alpha\cos\beta),$$

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$, $\mathrm{A} = \frac{d\Delta}{da}$, &c.,

$$\cos\alpha = \frac{p}{(p^2+q^2+r^2)^{\frac{1}{2}}}, \quad \cos\beta = \frac{q}{(p^2+q^2+r^2)^{\frac{1}{2}}}, \quad \cos\gamma = \frac{r}{(p^2+q^2+r^2)^{\frac{1}{2}}},$$

$$\frac{\Delta^2}{k^2} = [\mathrm{A}^2 + \mathrm{G}^2 + \mathrm{H}^2]\cos^2\alpha + \&c. + 2\,[\mathrm{GH} + \mathrm{F}\,(\mathrm{B}+\mathrm{C})]\cos\beta\cos\gamma + \&c.$$
$$= \mathrm{A}'\cos^2\alpha + \mathrm{B}'\cos^2\beta + \mathrm{C}'\cos^2\gamma + 2\mathrm{F}'\cos\beta\cos\gamma + 2\mathrm{G}'\cos\gamma\cos\alpha + 2\mathrm{H}'\cos\alpha\cos\beta,$$

$$\therefore \cos\theta = \frac{\mathrm{A}p^2 + \mathrm{B}q^2 + \mathrm{C}r^2 + 2\mathrm{F}qr + 2\mathrm{G}rp + 2\mathrm{H}pq}{\left\{p^2+q^2+r^2\right\}^{\frac{1}{2}}\left\{\mathrm{A}'p^2 + \mathrm{B}'q^2 + \mathrm{C}'r^2 + 2\mathrm{F}'qr + 2\mathrm{G}'rp + 2\mathrm{H}'pq\right\}^{\frac{1}{2}}}.$$

Note.—Equating to zero the numerators in the above values of $\cos\theta$, we get at once the known conditions for a point of inflexion on a plane curve and for a parabolic point on a surface, as we ought, $\theta$ being manifestly a right angle in both cases.

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II. *Solution by* J. J. Walker, M.A.

Let the equation to the curve, referred to two rectangular axes $y=0$, $x=0$, be $u=0$; and let the coordinates of any point on the curve relative

to parallel axes through a point $(x, y)$, also on the curve, be $(x', y')$; then the transformed equation will be

$$\frac{du}{dx}x' + \frac{du}{dy}y' + \frac{d^2u}{dx^2}x'^2 + 2\frac{d^2u}{dx\,dy}x'y' + \frac{d^2u}{dy^2}y'^2 + \dots\dots = 0 \dots\dots (1);$$

or, writing $\frac{du}{dx} = 2a$, $\frac{du}{dy} = 2b$, $\frac{d^2u}{dx^2} = c$, $\frac{d^2u}{dx\,dy} = d$, $\frac{d^2u}{dy^2} = e$,

for points very near $(xy)$ the curve will coincide with the conic

$$2ax' + 2by' + cx^2 + 2dxy + ey^2 = 0 \quad \dots\dots\dots\dots\dots\dots\dots (2);$$

and if $\phi$ be the angle between the normal and diameter through the new origin, it is readily found that $\tan\phi = \frac{(a^2-b^2)d - ab(c-e)}{a^2e + b^2c - 2abd}$. This diameter is the line which bisects the indefinitely near chord of the given curve which is parallel to the tangent at $(xy)$, so that the required angle is

$$\tan^{-1}\frac{\left\{\left(\frac{du}{dx}\right)^2 - \left(\frac{du}{dy}\right)^2\right\}\frac{d^2u}{dx\,dy} - \frac{du}{dx}\frac{du}{dy}\left(\frac{d^2u}{dx^2} - \frac{d^2u}{dy^2}\right)}{\left(\frac{du}{dx}\right)^2\frac{d^2u}{dy^2} + \left(\frac{du}{dy}\right)^2\frac{d^2u}{dx^2} - 2\frac{du}{dx}\frac{du}{dy}\frac{d^2u}{dx\,dy}}.$$

Similarly, if $\phi$ be the angle between the normal at $(xyz)$ to the surface $u=0$, and the line joining that point with the centre of the section of the indicatrix parallel and indefinitely near to the tangent plane at $(xyz)$,

$$\cos\phi = \frac{a^2\mathrm{D} + b^2\mathrm{E} + c^2\mathrm{F} + 2bc\mathrm{G} + 2ca\mathrm{H} + 2ab\mathrm{K}}{\left\{(a\mathrm{D} + b\mathrm{K} + c\mathrm{H})^2 + (a\mathrm{K} + b\mathrm{E} + c\mathrm{G})^2 + (a\mathrm{H} + b\mathrm{G} + c\mathrm{F})^2\right\}^{\frac{1}{2}}(a^2 + b^2 + c^2)^{\frac{1}{2}}}$$

where $a = \frac{du}{dx}$, $b = \frac{du}{dy}$, $c = \frac{du}{dz}$ and D, E, ... K are the partial differential coefficients with respect to $d, e, \dots k$ of $def + 2ghk - dg^2 - eh^2 - fk^2$, and

$$d = \frac{d^2u}{dx^2},\quad e = \frac{d^2u}{dy^2},\quad f = \frac{d^2u}{dz^2},\quad g = \frac{d^2u}{dydz},\quad h = \frac{d^2u}{dz\,dx},\quad k = \frac{d^2u}{dx\,dy}.$$

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**3678.** (Proposed by Professor Wolstenholme.)—An arithmetical, a geometrical, and an harmonical progression have each the same first and last terms and the same number of terms $(n)$; and $a_r$, $b_r$, $c_r$ denote the $r$th terms of each; prove that $a_r : b_r = b_{n-r+1} : c_{n-r+1}$, and thence that, if A, B, C be the continued products of all the terms of the three series respectively, $\mathrm{B}^2 = \mathrm{AC}$.

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*Solution by the* Rev. R. Harley, F.R.S.; W. Hoskins; *and others.*

Let $a$ be the first and $c$ the last term of each series; then, inserting the $n-2$ intervening terms, we have

for the arithmetical progression, $a,\ a + \frac{c-a}{n-1},\ a + 2\frac{c-a}{n-1},\ \dots\dots\ c$;

for the geometrical progression, $a,\ a\left(\frac{c}{a}\right)^{\frac{1}{n-1}},\ a\left(\frac{c}{a}\right)^{\frac{2}{n-1}},\ \dots\dots\ c$;

for the harmonical progression,

$$a, \quad \frac{ac(n-1)}{c(n-1)+(a-c)}, \quad \frac{ac(n-1)}{c(n-1)+2(a-c)}, \quad \ldots\ldots \; c.$$

Consequently we have

$$a_r = a + \frac{r-1}{n-1}(c-a) = \frac{a(n-r)+c(r-1)}{n-1},$$

$$b_r = a\left(\frac{c}{a}\right)^{\frac{r-1}{n-1}}, \quad b_{n-r+1} = a\left(\frac{c}{a}\right)^{\frac{n-r}{n-1}},$$

$$c_r = \frac{ac(n-1)}{c(n-1)+(r-1)(a-c)} = \frac{ac(n-1)}{c(n-r)+a(r-1)},$$

$$c_{n-r+1} = \frac{ac(n-1)}{a(n-r)+c(r-1)};$$

so that $\qquad a_r c_{n-r+1} = ac, \quad$ and $\quad b_r b_{n-r+1} = ac;$

therefore $\qquad a_r c_{n-r+1} = b_r b_{n-r+1},$

which establishes the first relation, viz.,

$$a_r : b_r = b_{n-r+1} : c_{n-r+1}.$$

The second may be deduced from the same equation thus:—Making $r = 1, 2, \ldots n$ successively, we have

$$a_1 c_n = b_1 b_n, \quad a_2 c_{n-1} = b_2 b_{n-1}, \quad \ldots\ldots \quad a_n c_1 = b_n b_1;$$

and since $\qquad A = a_1 a_2 \ldots a_n, \quad B = b_1 b_2 \ldots b_n, \quad C = c_1 c_2 \ldots c_n,$

therefore $\qquad AC = B^2.$

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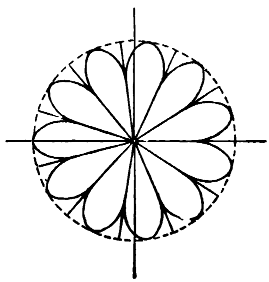
**3682.** (Proposed by the Editor.)—Find the envelope of the straight line which joins the extremities of the hands of a clock.

*Solution by the* Rev. W. H. Laverty, M.A.

1. To find the envelope when the hands are of equal length ($a$). Take the axis of $x$ as the noon-line; and suppose the hands to revolve backwards.



Let us find the equation to the reciprocal of this envelope. Reciprocating with respect to the circle whose radius is $a$, the locus is the intersection of the two lines

$$x \cos\alpha + y \sin\alpha = a,$$

$$x \cos 12\alpha + y \sin 12\alpha = a.$$

Let $\dfrac{x}{\cos\theta} = \dfrac{y}{\sin\theta} = r$; and eliminate $\alpha$, then we have $r = a \sec \dfrac{11\theta}{13}$.

If we trace this curve we shall find it to be as follows:

There will be 11 asymptotes, all tangents to the circle whose radius is $\frac{13}{11}a$; and inclined, in order, to the axis of $x$, at angles $\frac{13}{11}\cdot\frac{1}{2}\pi$, $\frac{17}{11}\cdot\frac{1}{2}\pi$, $\frac{21}{11}\cdot\frac{1}{2}\pi$, and so on. Every consecutive pair of these asymptotes will be asymptotes to one branch of the curve which will be shaped somewhat like half an hyperbola. There will thus be 11 branches, and no part of the curve will fall within the circle $r=a$. Now reciprocate back again, and the required envelope becomes a re-entering curve of 11 branches as in the figure.

2. When the hands are not of equal length, we shall have to eliminate $\alpha$ between the equations

$$x\cos\alpha + y\sin\alpha = a, \quad x\cos 12\alpha + y\sin 12\alpha = \frac{a^2}{b}.$$

The result will be $$11\theta = 12\cos^{-1}\tfrac{1}{r}a + \cos^{-1}\frac{a^2}{br}.$$

This will be similar to the other, having identically the same asymptotes, but the branches will not be quite so symmetrical. Reciprocating back we get a similar curve to the one drawn above.

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**3623.** (Proposed by the Rev. Dr. Booth, F.R.S.)—From a point T two tangents, making an angle $\phi$ with one another, are drawn to a conic; and also two focal vectors $r$ and $r_1$. From T a perpendicular $p$ is drawn on a focal chord passing through one of the points of contact: (1) prove that $rr_1\sin\phi = 2ap$, and (2) apply this formula to determine the radius of curvature of a conic.

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I. *Solution by* J. J. Walker, M.A.

Let O be centre, F, F′ foci, P, P′ points of contact of tangents from T; and let TO meet the chord of contact PP′ in Q: then

1. $\text{TPF} + \text{TPF}' = \frac{1}{2}(\text{PF} + \text{PF}')p = ap$, and similarly $\text{TP}'\text{F} + \text{TP}'\text{F}' = ap$,

Hence $2ap = \text{TPFP}' + \text{TPF}'\text{P}' = 2\text{TPP}' + \text{PFP}' + \text{PF}'\text{P}' = 2(\text{TPP}' + \text{OPP}')$
$= 2\text{TPP}' \times \text{OT} \div \text{QT} = rr_1\sin\phi,$

since, as is known, $rr_1 : \text{TP}\,.\,\text{TP}' = \text{OT} : \text{QT}$.

2. Now suppose T to approach indefinitely near to the curve, so that $\sin\phi$ becomes $\phi$, the angle of contingence, and equal to $\frac{2\text{TP}}{\rho}$, $\rho$ being the radius of curvature at P. The equation above gives therefore, in the limit,

$$\rho = \frac{rr_1\text{TP}}{ap} = \frac{rr_1}{a\sin\text{FPT}} = \frac{(rr)^{\frac{3}{2}}}{ab}.$$

What precedes relates to the central conics. In the case of the parabola, $r_1 = 2a$, so that the general equation becomes $r\sin\phi = p$; whence, if T approaches the curve without limit,

$$\rho = \frac{2r\text{TP}}{p} = \frac{2r}{\sin\text{FPT}} = 2\left(\frac{r^3}{m}\right),$$

$m$ being one-fourth of the parameter.

II. *Solution by the* PROPOSER.

If the foci of a conic section be defined as the points in which a plane section of a right cone is tonched by the inscribed spheres or sphere, it may easily be shown that the angle between any pair of tangents drawn in the plane of this section from the point T, is equal to the angle between the lines drawn from the same point T in the tangent plane to the cone to the points in which this tangent plane touches the inscribed spheres. As these lines so drawn are manifestly equal to the focal distances of the point T, seeing that they are two by two tangents to the same sphere, the proposition becomes evident.

NOTE.—It may be observed that a solution of this question by the ordinary methods of algebraical notation, whether projective or tangential, is exceedingly cumbersome and operose,—a pregnant instance of the advantages which the use of a variety of methods may afford.

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**3588.** (Proposed by G. O'HANLON.)—At a latitude $\theta$, on a given day in the year, at 12 o'clock mid-day, a mass $m$ weighs W; find the disturbing effect of the sun's attraction in altering the weight between noon and sunset.

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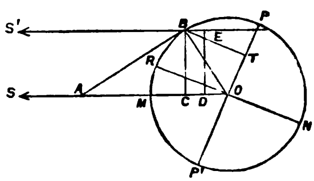
*Solution by the* PROPOSER; *and* G. S. CARR.




Through S the sun, and the earth, on the given day at noon, draw a plane cutting the earth in the great circle PMP'N, where PP' is the pole and RN the position of the equator on that day. Let B be the place on the earth, then ROB = latitude = $\theta$. Also S'BA = BAO = sun's altitude at noon = $\phi$. Draw BT parallel to RO and BC perpendicular to MO.

Now let B turn with radius BT and centre T perpendicularly to PP' through the hour-angle $\psi$, and we shall call its new position B'. From B' draw a perpendicular on BT meeting it in E, and draw ED parallel to BC. Let BO = $r$ and SO = R. Then BT = $r\cos\theta$, BE = $r\cos\theta \operatorname{vers}\psi$, CD = $r\cos\theta\operatorname{vers}\psi\sin(\phi+\theta)$, and CO = $r\sin\phi$;

$$\text{therefore}\quad \text{DO} = \text{CO}-\text{CD} = r\left\{\sin\phi - \cos\theta\operatorname{vers}\psi\sin(\phi+\theta)\right\};$$

$$\text{therefore}\quad \cos \text{B'OD} = \frac{\text{DO}}{r} = \sin\phi - \cos\theta\operatorname{vers}\psi\sin(\phi+\theta).$$

Let $\mu$ = a constant depending on the force of attraction; also let M be the mass of the sun, and let $f$ be the accelerative force due to the earth's attraction only; then the accelerative force at B is $f$, minus the difference between the attraction of the sun at B and at the centre of the earth, resolved in the direction OB, that is,

$$= f - \mu\text{M}\left\{\frac{1}{(\text{R}-\text{CO})^2} - \frac{1}{\text{R}^2}\right\}\sin\phi = f-\mu\text{M}\left\{\frac{1}{(\text{R}-r\sin\phi)^2}-\frac{1}{\text{R}^2}\right\}\sin\phi.$$

The acceleration at $B' = f - \mu M \left\{ \frac{1}{(R-DO)^2} - \frac{1}{R^2} \right\} \cos B'OD$

$$= f - \mu M \left\{ \frac{1}{\{R - r\sin\phi + r\cos\theta \operatorname{vers}\psi \sin(\phi+\theta)\}^2} - \frac{1}{R^2} \right\}$$
$$\times \left\{ \sin\phi - \cos\theta \operatorname{vers}\psi \sin(\phi+\theta) \right\}.$$

Now let us see, from this last formula, at what hour, $\psi$, the sole acceleration $f$ is manifested; clearly when either of the factors of $\mu M$ vanishes. Equating them separately to zero, we find from the two factors respectively

$$\operatorname{vers}\psi = \frac{a\sin\phi - 1 \pm 1}{a\cos\theta\sin(\phi+\theta)}, \quad \operatorname{vers}\psi = \frac{\sin\phi}{\cos\theta\sin(\phi+\theta)}, \quad \text{where } \frac{r}{R} = a.$$

If we take the positive sign in the first expression, it reduces to the second, and it is easily seen, as might have been suspected, that the formula gives the hour of sunset. If we take the negative sign, the versine is not possible unless $R < r$, which is absurd.

To eliminate $\mu$, we have $\mu \frac{4}{3}\pi s r = f$ at the surface of the earth at sunset; therefore $\mu = \frac{3f}{4\pi s r}$, where $s$ is the specific gravity of the earth.

It can be shown that the weight is a maximum at sunset and a minimum at noon; therefore, to find the difference of acceleration at noon and the hour-angle $\psi$, we subtract the first formula from the second. It is not easy to exhibit the result in any convenient form; but, if we put $m =$ the ratio of the sun's to the earth's radius, $b =$ the difference of the versine of the given hour-angle and the hour of sunset, and $S =$ the angle of sunset, then divide by $f$, we find the ratio of the loss of weight is

$$(ma)^3 n \sin^2\phi \left\{ \frac{2 - a\sin\phi}{(1 - a\sin\phi)^2} + \frac{b^2}{\operatorname{vers} S} \frac{2\operatorname{vers} S + ab\sin\phi}{(\operatorname{vers} S + ab\sin\phi)^2} \right\}.$$

For the earth $m = 110$, $n = \frac{1}{4}$, and $a = \frac{1}{23125}$; and taking the most favourable case for a high ratio, we find that a ton at the equator gains one grain from noon to sunset, and that the ratio of the times a pendulum would vibrate at noon and sunset is $\left(\frac{20000000}{20000001}\right)^{\frac{1}{2}}$.

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**3683.** (Proposed by B. Williamson, M.A.)—Reduce the following determinant to its simplest form,—

$$\begin{vmatrix} a_{11}a_{22} - a^2_{12}, & a_{11}a_{23} - a_{12}a_{13}, & \ldots\ldots & a_{11}a_{2n} - a_{12}a_{1n} \\ a_{11}a_{23} - a_{12}a_{13}, & a_{11}a_{33} - a^2_{13}, & \ldots\ldots & a_{11}a_{3n} - a_{13}a_{1n} \\ a_{11}a_{24} - a_{12}a_{14}, & a_{11}a_{34} - a_{13}a_{14}, & \ldots\ldots & a_{11}a_{4n} - a_{14}a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{11}a_{2n} - a_{12}a_{1n}, & a_{11}a_{3n} - a_{13}a_{1n}, & \ldots\ldots & a_{11}a_{nn} - a^2_{1n} \end{vmatrix}$$

I. *Solution by* Professor TOWNSEND, M.A., F.R.S.

Denoting by A the symmetrical determinant

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \cdots\cdots & a_{1n} \\ a_{12}, & a_{22}, & a_{23}, & \cdots\cdots & a_{2n} \\ a_{13}, & a_{23}, & a_{33}, & \cdots\cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots\cdots & \vdots \\ a_{1n}, & a_{2n}, & a_{3n}, & \cdots\cdots & a_{nn} \end{vmatrix}$$

and by $A_{11}$, $A_{12}$, $A_{13}$, $A_{14}$, &c., its several first minors corresponding to the several constituents $a_{11}$, $a_{12}$, $a_{13}$, $a_{14}$, &c., of its first row or column, the several constituents of each minor being supposed to remain in the order in which they are found after the erasure of the row and column containing the corresponding constituent of A; then, of the $2^{n-1}$ separate determinants into which the given symmetrical determinant $\Delta$ may be resolved, all but $n$ vanishing in consequence of the identity (common factors excluded) of two or more of their columns, and the $n$ that remain, expressed in terms of the above minors, being respectively

$$+a_{11}^{n-1}A_{11}, \quad -a_{11}^{n-2}a_{12}A_{12}, \quad +a_{11}^{n-2}a_{13}A_{13}, \quad -a_{11}^{n-2}a_{14}A_{14}, \quad +\text{\&c.};$$

therefore $\Delta = a_{11}^{n-2}\left[a_{11}A_{11}-a_{12}A_{12}+a_{13}A_{13}-a_{14}A_{14}+\text{\&c.}\right] = a_{11}^{n-2}A.$

II. *Solution by the* Rev. W. H. LAVERTY, M.A.

The determinant may evidently be split up into $2^{n-1}$ other determinants of the same order. Now if any one of these determinants contain less than $(n-2)$ of the columns whose factors are $a_{11}$, it will also contain at least two columns which (except for common factors) are identical. Therefore all such will vanish, and the only ones left will be the $(n-1)$ determinants which contain $(n-1)$ of the columns with factors $a_{11}$; and the single determinant which contains $n$ such columns; these will combine evidently into the determinant $a_{11}^{n-2}A$.

III. *Solution by the* PROPOSER.

The determinant in question is evidently equivalent to

$$\frac{1}{a_{11}}\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \cdots\cdots & a_{1n} \\ 0, & a_{11}a_{22}-a_{12}^2, & a_{11}a_{23}-a_{12}a_{13}, & \cdots\cdots & a_{11}a_{2n}-a_{12}a_{1n} \\ 0, & a_{11}a_{23}-a_{12}a_{13}, & a_{11}a_{33}-a_{13}^2, & \cdots\cdots & a_{11}a_{3n}-a_{13}a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & a_{11}a_{2n}-a_{12}a_{1n}, & a_{11}a_{3n}-a_{13}a_{1n}, & \cdots\cdots & a_{11}a_{nn}-a_{1n}^2 \end{vmatrix}$$

To find its value we multiply the first row by $a_{12}$, $a_{13}$, $\ldots a_{1n}$ in succession, and adding the first product to the second row, the next to the third row, and so on to the last; then observing that the terms in all the rows in the

new determinant (after the first row) contain $a_{11}$ as a factor, we obtain $a_{11}^{n-2}$ A for the value of the proposed determinant.

[These reductions are simpler than that given in Mr. WILLIAMSON'S *Differential Calculus*, p. 342. The determinant in question is of importance in connection with the general conditions for maxima and minima for any number of variables.]

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**3546.** (Proposed by R. TUCKER, M.A.)—The angles of a triangle are taken as the eccentric angles of three points P, Q, R on an ellipse; find the envelope of the locus of the centres of the circles about PQR (P being supposed at first fixed, and then to vary its position). Find also the locus of the centres of the circles which the circles touch when a point is fixed.

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*Solution by* S. WATSON; *the* PROPOSER; *and others.*

Let $\alpha, \beta, \gamma$ be the eccentric angles, so that $\alpha+\beta+\gamma=\pi$; then the coordinates of P, Q, R are $(a\cos\alpha,\ b\sin\alpha)$, $(a\cos\beta,\ b\sin\beta)$, and $(a\cos\gamma,\ b\sin\gamma)$; and the perpendiculars to PQ, QR at their middle points are easily found to be
$$by-ax\cot\tfrac{1}{2}\gamma=-c^2\cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}(\alpha-\beta),$$
and
$$by-ax\cot\tfrac{1}{2}\alpha=-c^2\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}(\beta-\gamma);$$
therefore $by=-c^2\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma$, and $ax=c^2\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma$;
therefore
$$by\tan\tfrac{1}{2}\alpha+ax=-c^2\sin^2\tfrac{1}{2}\alpha \quad\text{.......................... (1)};$$
hence the locus of the centres when P is fixed is a straight line.

Differentiate (1) with respect to $\alpha$; then we have

$2by=-c^2\sin\alpha\,(1+\cos\alpha)$, and therefore $2ax=c^2\cos\alpha\,(1-\cos\alpha)$.

Substituting the value of $\cos\alpha$ derived from the latter in the square of the first, and clearing of radicals, we have
$$\left\{4\,(b^2y^2+a^2x^2)-\tfrac{1}{4}(c^2+20ax)\,c^2\right\}^2=\tfrac{1}{4}c^2\,(c^2-ax)^3 \quad\text{............(A)},$$
the equation of the envelope of (1), when P varies its position.

Again, the equation of the circle about PQR is
$$\left(x-\frac{c^2}{a}\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma\right)^2+\left(y+\frac{c^2}{q}\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma\right)^2$$
$$=\left(a\cos\alpha-\frac{c^2}{a}\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma\right)^2+\left(b\sin\alpha+\frac{c^2}{b^2}\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma\right)^2,$$
or
$$x^2+y^2-c^2\sin\tfrac{1}{2}\alpha\left(\frac{x}{a}-\cos\alpha\right)\left\{\cos\tfrac{1}{2}(\beta-\gamma)-\sin\tfrac{1}{2}\alpha\right\}$$
$$+c^2\cos\tfrac{1}{2}\alpha\left(\frac{y}{b}-\sin\alpha\right)\left\{\cos\tfrac{1}{2}(\beta-\gamma)+\sin\tfrac{1}{2}\alpha\right\}=a^2\cos^2\alpha+b^2\sin^2\alpha \ \ldots\ (2).$$

Differentiating with respect to $\beta$ and $\gamma$, on the condition $\beta+\gamma=$ a constant, the result gives $\beta=\gamma$, and (2) becomes
$$x^2+y^2-\frac{c^2x}{a}\sin\tfrac{1}{2}\alpha\,(1-\sin\tfrac{1}{2}\alpha)+\frac{c^2y}{b}\cos\tfrac{1}{2}\alpha\,(1+\sin\tfrac{1}{2}\alpha)$$
$$=a^2+c^2\sin\tfrac{1}{2}\alpha\,(1-\sin\tfrac{1}{2}\alpha)\ \text{......(3)}.$$
Hence, when P is fixed, the circles about PQR always touch the circle (3).

If $x, y$ be now the coordinates of the centre of (3), we have

$$2ax = c^2 \sin \tfrac{1}{2}\alpha (1 - \sin \tfrac{1}{2}\alpha), \quad 2by = -c^2 \cos \tfrac{1}{2}\alpha (1 + \sin \tfrac{1}{2}\alpha);$$

and the elimination of $\alpha$ gives the same result (A) as above. Therefore, when P varies, the envelope of (1) and the locus of the centres of (3) are identical.

[Mr. TUCKER obtains equation (1) in the following manner:—

If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of four points on an ellipse which are also on a circle, then $\alpha + \beta + \gamma + \delta = 0$ or $\sin \pi$, and, by hypothesis, $\alpha + \beta + \gamma = \pi$; therefore all the circles pass through a fixed point, viz., the extremity of the major axis. Hence, if $\alpha$ is fixed, the locus of centres will be a straight line perpendicular to that joining ($\delta$) and ($\alpha$) through its middle point, that is, the line whose equation is

$$2by \sin \tfrac{1}{2}\alpha + 2ax \cos \tfrac{1}{2}\alpha = -c^2 \sin \alpha \sin \tfrac{1}{2}\alpha \quad \ldots\ldots\ldots\ldots\ldots (1).]$$

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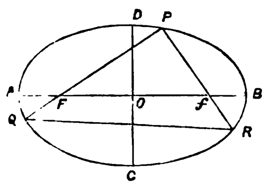
**3547.** (Proposed by S. WATSON.)—From any point P in the curve of an ellipse two lines are drawn through the foci F, $f$, meeting the curve again in Q and R. Find (1) the average area of the triangle PQR; (2) the locus of the pole of QR; (3) the envelope of QR; (4) the chance that QR shall be less than F$f$.

*Solution by the* PROPOSER.




1. Take the principal diameters AB, CD for axes, and denote the points P, Q, R by $(x', y')$, $(x_1, y_1)$, $(x_2, y_2)$. Then, putting $m^2 = a^2 + c^2$, we have

$$\text{PF} (= a + ex') : \text{QF} (= a - ex_1) = c + x' : -(c + x_1);$$

therefore $$x_1 = -\frac{2a^2c + m^2x'}{m^2 + 2cx'} \quad \ldots\ldots\ldots (1).$$

Similarly $$x_2 = \frac{2a^2c - m^2x'}{m^2 - 2cx'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

$$\text{PF} : \text{PQ} = c + x' : x' - x_1, \quad \text{and} \quad \text{P}f : \text{PR} = c - x' : x_2 - x';$$

therefore $\triangle \text{PF}f (= cy') : \triangle \text{PQR} = c^2 - x'^2 : (x' - x_1)(x_2 - x')$;

therefore $$\triangle \text{PQR} = \frac{(a^4 - c^2x'^2)\, cy'}{m^4 - 4c^2x'^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Now an element of the curve at P is

$$\frac{(a^4y'^2 + b^4x'^2)^{\frac{1}{2}}}{a^2y'}\, dx' = \frac{b\,(a^4 - c^2x'^2)^{\frac{1}{2}}}{a^2y'}\, dx',$$

and if $l$ be the length of a quadrantal arc of the ellipse, the average area (A) of the triangle PQR is

$$\frac{bc}{la^2}\int_0^a \frac{(a^4 - c^2x'^2)^{\frac{3}{2}}}{m^4 - 4c^2x'^2}\, dx',$$

which, by putting $cx' = a^2 \sin\phi$, becomes

$$= \frac{a^6 b}{l}\int_0^{\sin^{-1}e} \frac{\cos^4\phi\, d\phi}{m^4 - 4a^4\sin^2\phi}$$

$$= \frac{a^2 b}{l}\int_0^{\sin^{-1}e} d\phi \left\{ -\tfrac{1}{4}\sin^2\phi + \tfrac{1}{2} - \tfrac{1}{16}(1+e^2)^2 + \frac{[\frac{1}{4}(1+e^2)^2 - 1]^2}{(1+e^2)^2 - 4\sin^2\phi} \right\}$$

$$= \frac{a^2 b}{16 l}\left\{ [6 - (1+e^2)^2]\sin^{-1}e + 2e(1-e^2)^{\frac{1}{2}} \right.$$
$$\left. + \frac{[(1+e^2)^2 - 4]^{\frac{3}{2}}}{1+e^2} \tan^{-1}\left[\frac{\{(1+e^2)^2 - 4\}^{\frac{1}{2}} e}{(1+e^2)(1-e^2)^{\frac{1}{2}}}\right] \right\}.$$

2. In the same manner that (1), (2) were found, we have

$$y_1 = -\frac{b^2 y'}{m^2 + 2cx'}, \quad \text{and} \quad y_2 = -\frac{b^2 y'}{m^2 - 2cx'} \quad\ldots\ldots\ldots\ldots\ldots (4, 5).$$

The equations of tangents at Q and R are

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1, \quad \text{and} \quad \frac{x_2 x}{a^2} + \frac{y_2 y}{b^2} = 1;$$

and substituting from (1), (2), and (4), (5), the results easily give

$$x' = -x, \quad y' = \frac{m^2(x^2 - a^2)}{a^2 y},$$

and putting these in $\qquad \dfrac{x'^2}{a^2} + \dfrac{y'^2}{b^2} = 1,$

we have $\qquad m^4 x^2 + a^2 b^2 y^2 = a^2 m^4,$

hence the locus of the pole of QR is an ellipse.

3. The equation of QR is $\quad (y_1 - y_2)x - (x_1 - x_2)y = x_2 y_1 - x_1 y_2,$

which, by substitution as before, becomes

$$b^4 x' x + a^2 m^2 y' y + a^2 b^4 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and this must be subject to the condition $\quad b^2 x'^2 + a^2 y'^2 = a^2 b^2 \ldots\ldots\ldots\ldots (7);$

therefore $\qquad \dfrac{dy'}{dx'} = -\dfrac{b^4 x}{a^2 m^2 y} = -\dfrac{b^2 x'}{a^2 y'}, \quad$ or $\quad b^2 x y' = m^2 y x' \ldots\ldots\ldots\ldots\ldots (8).$

Hence the values of $x'$, $y'$ derived from (6) and (8), substituted in (7), gives

$$b^6 x^2 + a^2 m^4 y^2 = a^2 b^6,$$

and the envelope of QR is an ellipse also.

4. In order that QR may be less than F$f$, we must have

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 < 4c^2,$$

which, by substitution, becomes

$$4(a^2 - x'^2)\left\{a^4 m^4 - (a^2 m^4 - b^6)x'^2\right\} < a^2(m^4 - 4c^2 x'^2)^2,$$

from which $x'$ can be found; thence that portion of the curve on which P must lie, and thus the required chance may be found.

**3603.** (Proposed by the Rev. G. H. HOPKINS, M.A.)—1. If $P_1, P_2, P_3, P_4$ be the points of intersection of a circle with an ellipse, and $\phi_1, \phi_2, \phi_3, \phi$. be the excentric angles of those points, then, $\phi_1+\phi_2+\phi_3+\phi_4 = 0$, or $n\pi$.

2. If P be a point on an ellipse the excentric angle of which is $\phi$, and Q be the corresponding point on the evolute, then

$$x = \frac{a^2-b^2}{a}\cos^3\phi, \quad y = -\frac{a^2-b^2}{b}\sin^3\phi.$$

3. If P be any point on an ellipse, and Q the other extremity of the common chord of curvature; then at Q, $x = a\cos 3\phi$, $y = -b\sin 3\phi$, $\phi$ being the excentric angle of P.

4. If circles of curvature be drawn at the four points where an ellipse is intersected by a circle, the extremities of the common chords of their circles with the ellipse will lie upon another circle.

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*Solution by the* PROPOSER.

1. The equation to the line passing through the middle point of $P_1P_2$, and at right angles to it, will be

$$by\cos\tfrac{1}{2}(\phi_1+\phi_2)-ax\sin\tfrac{1}{2}(\phi_1+\phi_2) = \tfrac{1}{2}(b^2-a^2)\sin(\phi_1+\phi_2)\cos\tfrac{1}{2}(\phi_1-\phi_2).$$

A similar line through the middle point of $P_2P_3$ will be

$$by\cos\tfrac{1}{2}(\phi_2+\phi_3)-ax\sin\tfrac{1}{2}(\phi_2+\phi_3) = \tfrac{1}{2}(b^2-a^2)\sin(\phi_2+\phi_3)\cos\tfrac{1}{2}(\phi_2-\phi_3).$$

These intersect in the point $(x, y)$ given by

$$\left.\begin{aligned} x &= \frac{a^2-b^2}{a}\cos\tfrac{1}{2}(\phi_1+\phi_2)\cos\tfrac{1}{2}(\phi_2+\phi_3)\cos\tfrac{1}{2}(\phi_3+\phi_1) \\ z &= -\frac{a^2-b^2}{b}\sin\tfrac{1}{2}(\phi_1+\phi_2)\sin\tfrac{1}{2}(\phi_2+\phi_3)\sin\tfrac{1}{2}(\phi_3+\phi_1) \end{aligned}\right\} \text{......(A)},$$

which will be the centre of the circle passing through $P_1, P_2, P_3$.

Similarly, the centre of the circle through $P_2, P_3, P_4$ will be given by

$$\left.\begin{aligned} x &= \frac{a^2-b^2}{a}\cos\tfrac{1}{2}(\phi_2+\phi_3)\cos\tfrac{1}{2}(\phi_3+\phi_4)\cos\tfrac{1}{2}(\phi_4+\phi_2) \\ z &= -\frac{a^2-b^2}{a}\sin\tfrac{1}{2}(\phi_2+\phi_3)\sin\tfrac{1}{2}(\phi_3+\phi_4)\sin\tfrac{1}{2}(\phi_4+\phi_2) \end{aligned}\right\} \text{......(B)}.$$

The points A and B are identical; therefore

$$\cos\tfrac{1}{2}(\phi_2+\phi_3)\cos\tfrac{1}{2}(\phi_3+\phi_4)\cos\tfrac{1}{2}(\phi_4+\phi_2)$$
$$= \cos\tfrac{1}{2}(\phi_1+\phi_2)\cos\tfrac{1}{2}(\phi_2+\phi_3)\cos\tfrac{1}{2}(\phi_3+\phi_1),$$

from which $\qquad 2\phi_1+\phi_2+\phi_3 = 2n\pi\pm(\phi_2+\phi_3+\phi_4);$

therefore $\qquad 2(\phi_1+\phi_2+\phi_3+\phi_4) = 2n\pi \quad\text{and}\quad \phi_1 = \phi_4.$

The second of these makes the two points $P_1$ and $P_4$ coincide; the relation $\phi_1+\phi_2+\phi_3+\phi_4 = n\pi$ is that which holds when the four points lie upon an ellipse and circle.

2. When three of the points coincide, the circle is the circle of curvature. The relations A then become

$$x = \frac{a^2-b^2}{a}\cos^3\phi, \quad y = -\frac{a^2-b^2}{b}\sin^3\phi.$$

3. Also $\phi_4+3\phi = n\pi$ when the three points coincide; therefore $n\pi-3\phi$ is the excentric angle of the point where the circle of curvature cuts the ellipse.

4. If $\phi_1, \phi_2, \phi_3, \phi_4$ be the four points; then $n_1\pi-3\phi_1$, $n_2\pi-3\phi_2$, $n_3\pi-3\phi_3$, $n_4\pi-3\phi_4$ are the excentric angles of the points where the circles of curvature at $\phi_1, \phi_2, \phi_3, \phi_4$ again meet the ellipse, and

$$(n_1+n_2+n_3+n_4)\,\pi-3\phi_1-3\phi_2-3\phi_3-3\phi_4 = -3\,(\phi_1+\phi_2+\phi_3+\phi_4) = 0;$$

therefore a circle will pass through these points.

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**3564.** (Proposed by Professor CAYLEY.)—To determine the least circle enclosing three given points.

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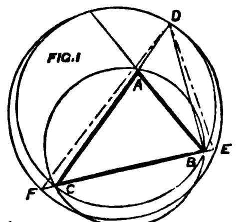
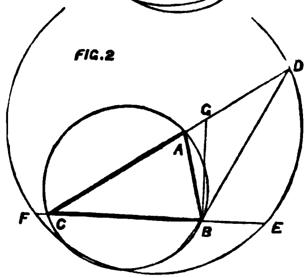
I. *Solution by* J. J. WALKER, M.A.




Call the points A, B, C; then, if one of the angles of the triangle ABC should be not less than a right angle, it is evident that the circle described on the opposite side as diameter will be the least circle enclosing the given points; for any other circle must have a chord, which is not a diameter, equal to or greater than that side.

2. If each angle of the triangle ABC be less than a right angle, consider a circle cutting CA produced in D, and CB produced in E, and BC produced in F; and in the first place (Fig. 1) suppose the angle CBD to be less than a right angle, and a circle to be described through D, C, B. This circle must be less than the circle DEF, their diameters being $\frac{\text{DB}}{\sin \text{DCB}}$, $\frac{\text{DE}}{\sin \text{DFE}}$ respectively, and DB being less than DE, while $\angle$DCB is greater than $\angle$DFE, and both are acute. But the circle ABC is less than any circle passing through B, C, and a point, such as D, in one of the sides BA or CA produced, the diameters being $\frac{\text{BC}}{\sin \text{BAC}}$ and $\frac{\text{BC}}{\sin \text{BDC}}$ respectively. In the second place (Fig. 2), if $\angle$CBD should not be less than a right angle, it is plain that the circle described on that part (CG) of CD as diameter, which subtends a right angle at B, would be less than the circle DEF, which has a chord greater than CD.

It follows, therefore, that if all the angles of the triangle ABC are acute, the least circle which will enclose the three points A, B, C is the circle passing through them.

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II. *Solution by* G. S. CARR.

The circle having any side of the triangle for its chord is least when

that side is a diameter. If the opposite angle falls without this circle, the circle must be *increased* to contain it. (Euc. III. 51.) Therefore, if each angle be acute, the circumscribing circle is the least enclosing the triangle, and if any angle be not acute, the opposite side is the diameter of the least circle.

III. *Solution by the* Rev. W. H. Laverty, M.A.

Let A $(x_1, y_1)$, B $(x_2, y_2)$, C $(x_3, y_3)$ be the three points; then it is plain that two of them (say A and B) must be on the circumference; hence we have

$$(x_1-\alpha)^2+(y_1-\beta)^2=r^2,\ \ (x_2-\alpha)^2+(y_2-\beta)^2=r^2,\ \ (x_3-\alpha)^2+(y_3-\beta)^2=r^2-k^2,$$

where $r$ is a minimum; therefore, differentiating with respect to $\alpha, \beta, k, r$, and equating $dr$ to zero, and eliminating, we get

$$k\left\{(x_1-\alpha)(y_2-\beta)-(x_2-\alpha)(y_1-\beta)\right\}=0;$$

therefore either $(x_1-\alpha)(y_2-\beta)-(x_2-\alpha)(y_1-\beta)=0\ \ldots(1)$, or $k=0\ \ldots(2)$.

(1) makes AB a diameter, which will happen when the angle C is not less than $\frac{1}{2}\pi$.

(2) makes C a point on the circle, which will happen when the angle C is not greater than $\frac{1}{2}\pi$.

**3628.** (Proposed by Professor Townsend, M.A., F.R.S.)—If A, B, C, D be the four values of the function $\left\{\sin s \sin(s-a) \sin(s-b) \sin(s-c)\right\}^{\frac{1}{2}}$ for the four triangles determined on the surface of a sphere by four radii parallel to any system of four equilibrating forces P, Q, R, S in space; prove that P : Q : R : S = A : B : C : D.

*Solution by* J. J. Walker, M.A.

Let the four radii meet the sphere in the four points $p, q, r, s$ respectively; and let the arc $rq=a$, $qs=b$, $sr=c$, $pq=d$, $pr=e$, $ps=f$. Then if the four forces P, Q, R, S are in equilibrium, each will be equal and opposite to the sum of the components of the other three along its direction

viz.,

$$\begin{aligned}
&P + Q\cos d + R\cos e + S\cos f = 0 \ \ldots\ldots\ldots\ldots (1),\\
&P\cos d + Q + R\cos a + S\cos b = 0 \ \ldots\ldots\ldots\ldots (2),\\
&P\cos e + Q\cos a + R + S\cos c = 0 \ \ldots\ldots\ldots\ldots (3),\\
&P\cos f + Q\cos b + R\cos c + S = 0 \ \ldots\ldots\ldots\ldots (4).
\end{aligned}$$

Eliminating P, Q, R, S, there results, writing for shortness $a' \ldots f'$ instead of $\cos a \ldots \cos f$,

$$\Delta = 1-(a'^2+b'^2+c'^2+d'^2+e'^2+f'^2)+a'^2f'^2+b'^2e'^2+c'^2d'^2$$
$$+2(a'b'c'+a'd'e'+c'e'f'+b'd'f')+2(a'c'd'f'+b'c'd'e'+a'b'e'f')=0\ldots(5).$$

Again, eliminating R, S from (2), (3), (4), there results $PE^2=QA^2$,

where $E^2=-(1-c'^2)d'+(a'-b'c')e'+f'(b'-a'c')$,

and $A^2=1-a'^2-b'^2-c'^2+2a'b'c'=\sin s\sin(s-a)\sin(s-b)\sin(s-c)$.

Now if $B^2 = 1 - c'^2 - e'^2 - f'^2 + 2c'e'f'$, it is easily verified that

$$E^4 \equiv A^2B^2 - (1-c^2)\,\Delta = A^2B^2,$$

in virtue of (5); hence $P : Q = A : B$; therefore, &c.

[Mr. TOWNSEND solves the question by supposing the four forces P, Q, R, S to be transferred parallel to themselves to the centre O of the sphere; then, since the four parallelepipeds determined by their four groups of three are manifestly equal, therefore $Q.R.S.A = R.S.P.B = S.P.Q.C = P.Q.R.D$, whence the theorem follows at once.]

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**3632.** (Proposed by C. TAYLOR, M.A.)—If $a, b, c, d$ be tangents to a circle, prove that the angle between the bisectors of the angles $(a, b)$, $(c, d)$ is equal to half the sum or difference of the angles $(a, c)$, $(b, d)$.

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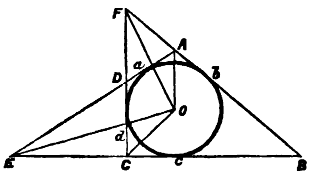
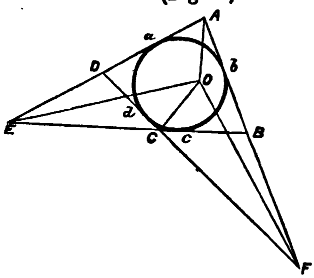
I. *Solution by* A. B. EVANS, M.A.; T. MITCHESON, B.A.; *and others.*

(Fig. 1.)




(Fig. 2.)




Let $A = \angle(a, b)$, $B = \angle(b, c)$, $C = \angle(c, d)$, $D = \angle(d, a)$, $E = \angle(a, c)$ $F = \angle(b, d)$, $\phi =$ the angle between the bisectors of the angles $(a, b)$, $(c, d)$. The bisectors of the angles A, C, E, F pass through O the centre of the circle. Then in Fig. 1 we have

$$AOC = \pi - \phi = B + \tfrac{1}{2}A + \tfrac{1}{2}C,$$
$$EOF = D - \tfrac{1}{2}E - \tfrac{1}{2}F,$$
$$AOF = \tfrac{1}{2}A - \tfrac{1}{2}F,$$
$$COE = \tfrac{1}{2}C - \tfrac{1}{2}E.$$

$$\therefore\quad AOC + EOF + AOF + COE = 2AOC = 2\pi - 2\phi = A + B + C + D - E - F;$$

therefore $\phi = \tfrac{1}{2}(E + F)$.

And in Fig. 2 we have

$$AOC = \pi - \phi = \tfrac{1}{2}A + F + \tfrac{1}{2}C + (ECD = \pi - C) = \tfrac{1}{2}A + F - \tfrac{1}{2}C + \pi,$$

$EOC = \tfrac{1}{2}C + \tfrac{1}{2}E$, $EOA = \pi - \tfrac{1}{2}A - \tfrac{1}{2}E$.

Therefore $AOC + EOC + EOA = 2AOC = 2\pi - 2\phi = 2\pi + F - E$;

therefore $\phi = \tfrac{1}{2}(E - F)$.

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II. *Solution by the* PROPOSER.

In Fig. 1 equate the angles of the two quadrilaterals AOCE, AOCF to eight right angles, and subtract the angles at A, C, whereof each pair make two right angles; then there remains $2AOC + E + F = 2\pi$.

In Fig. 2 equate the four angles of AOCE to those of AOCF, and subtract the equal angles at A, C; then there remains $\angle \mathrm{AOC} + \mathrm{E} = 2\pi - \mathrm{AOC} + \mathrm{F}$. Hence the acute angle between AO, CO is equal to $\tfrac{1}{2}(\mathrm{E} \pm \mathrm{F})$.

[The problem will be recognised under the following form:—the angle between two tangents to a conic is equal to half the sum or difference of the angles which their chord of contact subtends at the foci.]

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**3477.** (Proposed by T. Mitchesson, B.A.)—PQ is the vertical diameter, and PR a tangent to the circle PSTQ whose centre is O; and RSO, RTQ are straight lines. If $t$, $t_1$, $t_2$ be the respective times of a body falling from rest down PQ, RS, RT; prove that

$$\frac{\mathrm{RT}}{\mathrm{RS}} = \frac{4t_2^2}{\left\{(8t_1^2+t^2)^{\frac12}-t\right\}(t_2^2+t^2)^{\frac12}}.$$

*Solution by* Stephen Watson.

Put $\mathrm{PQ}=2a$, $\angle\mathrm{POR}=\theta$, $\angle\mathrm{PQR}=\phi$; then we easily find

$$\mathrm{RS} = \frac{a(1-\cos\theta)}{\cos\theta},\quad \mathrm{RT} = \frac{2a\sin^2\phi}{\cos\phi};$$

hence $t^2 : t_1^2 : t_2^2 = 2a : \dfrac{\mathrm{RS}}{\cos\theta} : \dfrac{\mathrm{RT}}{\cos\phi} = 2a : \dfrac{a(1-\cos\theta)}{\cos^2\theta} : \dfrac{2a\sin^2\phi}{\cos^2\phi}$;

therefore $\quad \cos\theta = \dfrac{t\left\{(8t_1^2+t^2)-t\right\}}{4t_1^2},\quad \cos\phi = \dfrac{t}{(t_2^2+t^2)^{\frac12}},$

and $\quad \dfrac{\mathrm{RT}}{\mathrm{RS}} = \dfrac{t_2^2\cos\phi}{t_1^2\cos\theta} = \dfrac{4t_2^2}{\left\{(8t_1^2+t^2)^{\frac12}-t\right\}(t_2^2+t^2)^{\frac12}}.$

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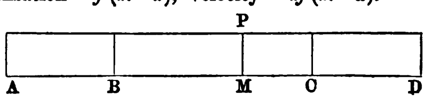
**3689.** (Proposed by R. Moon, M.A.)—A cylindrical bar rests upon a perfectly smooth table. A wave of condensation being propagated along the bar in a direction parallel to its axis, explain the nature of the reflection which takes place when the wave reaches the extremity of the bar.

*Solution by the* Proposer.

Suppose the condensation to be symmetrical with respect to its extremities, and to have a single maximum. It may be represented in the ordinary notation by condensation $= f(at-x)$, velocity $= af(at-x)$.

Let AB indicate the original position of the wave, CD its position when it reaches the opposite extremity of the bar.



The effect of the disturbance on any intermediate stratum, as PM, will be after a certain interval to communicate to it a velocity, gradually increasing from 0 up to a maximum, and thence gradually diminishing to 0; under the influence of which the stratum will move through a space equal to the amplitude of the wave, and will then be again at rest. During the motion there will always be a pressure on the left side of PM, urging it to the right; and one on the right urging it to the left. During the first half of the motion, the former will predominate; during the second half, the latter; that is, the existence of the pressure on the right side of PM during the first half of the motion makes the velocity less than it would otherwise be, while during the second half it actually diminishes it at each instant, till at last it finally destroys it.

But when the disturbance reaches the terminal stratum of the bar there will be a total absence of pressure upon the stratum from the right. Hence if no new principle were brought into play, the velocity communicated to the terminal stratum would never undergo diminution and separation from the other portions of the bar, which, it has been seen, have returned to rest, must infallibly take place.

This result will be prevented by the tension of the bar, by which the final stratum will drag after it the stratum next to it, thus diminishing its own velocity.

The same going on between stratum and stratum as we recede from D towards A, it is easy to see that a wave will be reflected backwards, in which the particle motion will take place in the same direction as before, that is, from left to right; the wave in this case, however, being one of rarefaction, since it is only in the case of such a wave that tension of the bar can occur.

When the reflected wave reaches A, a second reflection will take place towards D, the reflected wave in this case being a condensation as at first.

We may thus see how a blow administered under the circumstances supposed at one extremity of the bar in the direction of its axis, will drive the bar continually in that direction, and with the same average rate of progress as if the blow were bestowed on the same mass shrunk into an infinitely small ball; there being this difference in the particle motion in the two cases, that, in the former, supposing the bar long enough, the motion would be intermittent, at times wholly ceasing, and during its continuance ever varying; while in the latter case it would be uniform.

The progression of the bars, in fact, will be effected by a snail-like motion, the impulse given to the rear being first propagated forwards by pushing, and then again backwards by pulling, the particle motion throughout being in the same direction.

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**3482.** (Proposed by Rev. Dr. Booth, F.R.S.)—Eliminate $\theta$ between the equations $$m = \frac{a \sin\theta \cos^2\theta}{(a\cos\theta + b)^2} \qquad n = \frac{a\cos^3\theta + b}{(a\cos\theta + b)^2}.$$

*Solution by* Stephen Watson.

Put $a\cos\theta + b = r$, $m = \dfrac{u}{a}$, $n = \dfrac{v}{a}$, and for shortness' sake let

$$u^2+v^2-1=\mathrm{L},\quad 3b+av=\mathrm{M},\quad b^2-a^2=\mathrm{N};$$

then $\qquad u^2r^4=a^4\cos^4\theta\,(1-\cos^2\theta)=(r-b)^4-(vr^2-ab)^2;$

therefore $\qquad \mathrm{L}r^4+4br^3-2b\mathrm{M}r^2+4b^3r-b^2\mathrm{N}=0 \quad\text{.....................}\quad (1).$

Also $\quad avr^2=(r-b)^3+a^2b;\quad$ therefore $\quad r^3-\mathrm{M}r^2+3b^2r-b\mathrm{N}=0 \;......\; (2),$

and $(1)-b\,(2)$, gives $\qquad \mathrm{L}r^3+3br^2-b\mathrm{M}r+b^3=0 \quad\text{...........................}\quad (3).$

Eliminating $r$ between (2) and (3), we have

$$\left\{(b^2+\mathrm{LN})^2-(3b+\mathrm{LM})(3b^3-\mathrm{MN})\right\}^2$$
$$=b\left\{(b^2+\mathrm{LN})(b\mathrm{M}-3\mathrm{N})-(\mathrm{M}+3b\mathrm{L})(3b^3-\mathrm{MN})\right\}$$
$$\times\left\{(b^2+\mathrm{LN})(\mathrm{M}+3b\mathrm{L})-(3b+\mathrm{LM})(b\mathrm{M}-3\mathrm{N})\right\}.$$

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**3670.** (Proposed by the EDITOR.)—Can any value of $x$ be found which will make $927x^2-123x+413$ a rational square?

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*Solution by* MATTHEW COLLINS, B.A.

1. When $x$ is an integer, it is plain that

$$927x^2-123x+413=3\,(309x^2-41x+137)+2$$

can never be a square; since it is of the form $3\mathrm{N}+2$.

2. When $x$ is a fraction

$$927x^2-123x+413=103\,(3x)^2-41\,(3x)+413$$

cannot be equal to a rational square number $\mathrm{X}^2$; for if so, we should have

$$4\,(103)\,\mathrm{X}^2-23\,(293)\,25=\left\{206\,(3x)-41\right\}^2=\square,$$

or $$103\,(\tfrac{2}{5}\mathrm{X})^2-23\,(293)=\square.$$

Now if $\frac{2}{5}\mathrm{X}=\frac{y}{z}$, a fraction in its lowest terms, we should then have

$$103y^2-23\,(293)\,z^2=\square=w^2,$$

and to have this possible, $y$ could not be divisible by 23; for if so, then $z$, which is prime to $y$, could not be divisible by 23; so then $w^2$ would be divisible by 23, but not by $23^2$, which is impossible; and as $y$ is therefore prime to 23, we can therefore have $w=ny-23v$, $n$ and $v$ being integers, and the quantity $103y^2-23(293)z^2=w^2=(ny-23v)^2$, giving $\frac{1}{23}(n^2-11)\,y^2$ = the integer $4y^2-293z^2+2nyv-23v^2$, which is plainly impossible, since neither $y$ nor $n^2-11$ is ever divisible by 23.

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**3685.** (Proposed by J. W. L. GLAISHER, B.A., F.R.A.S.)—Prove that

$$u=x^{-3i}\text{ coefficient of }h^i\text{ in }\quad(1+hx^2)^{i-\frac{1}{2}}\left(c_1e^{\frac{qx}{(1+hx^2)^{\frac{1}{2}}}}+c_2e^{-\frac{qx}{(1+hx^2)^{\frac{1}{2}}}}\right)$$

is the complete integral of the equation

$$\frac{d^2u}{dx^2} - \frac{i(i+1)}{x^2}u - q^2u = 0.$$

*Solution by the* PROPOSER.

It was shown by BOOLE (*Phil. Trans.*, 1844, and *Diff. Equations*, p. 425) that the solution of the proposed differential equation could be written in the form

$$u = x^{-i-1}\left(x^3\frac{d}{dx}\right)\frac{c_1e^{qx}+c_2e^{-qx}}{x^{2i-1}}.$$

Take $x = y^{-\frac{1}{2}}$, and we find

$$u = y^{\frac{1}{2}(i+1)}\left(\frac{d}{dy}\right)^i \frac{c_1e^{\frac{q}{\sqrt{y}}}+c_2e^{-\frac{q}{\sqrt{y}}}}{y^{-i+\frac{1}{2}}}$$

$$= y^{\frac{1}{2}(i+1)} \text{ coefficient of } h^i \text{ in } \frac{c_1e^{\frac{q}{\sqrt{(y+h)}}}+c_2e^{-\frac{q}{\sqrt{(y+h)}}}}{(y+h)^{-i+\frac{1}{2}}};$$

and the result in the question follows at once by writing $x^{-2}$ for $y$.

3694. (Proposed by J. HOPKINSON, D.Sc., B.A.)—An elastic rod is capable of turning about its axis of figure; to its extremities are attached two discs whose moments of inertia are $Mk^2$, $M'k'^2$ about the rod; the system revolves with mean angular velocity $n$, and the discs M and M′ are acted on by couples $L(1+\cos p\theta)$ and $-L$, where L is small and $\theta, \theta'$ are the angles turned through by the disc. Determine the motion of the system, and show that if the rod can only twist to a small extent without breaking, it must break, no matter how small L may be; a certain condition subsisting which connects the moments of inertia, the stiffness of the rod, and $p$.

*Solution by the* PROPOSER.

When the whole twist in the rod is $\alpha$, let $K\alpha$ be the couple at any point, K being a constant. The mean value of $\theta-\theta'$ must then be $\frac{L}{K}$, and the equations of motion

$$Mk^2\frac{d^2\theta}{dt^2} = L(1+\cos p\theta) - K(\theta-\theta'),\quad M'k'^2\frac{d^2\theta'}{dt^2} = -L + K(\theta-\theta') \ldots (1).$$

Supposing we were to neglect L, the time of vibration would be

$$2\pi\left\{K\left(\frac{1}{Mk^2}+\frac{1}{M'k'^2}\right)\right\}^{-\frac{1}{2}}, \text{ which is independent of } n.$$

Now we may put $\theta = nt + \frac{L}{K} + \phi,\quad \theta' = nt + \phi',$

when $\phi$ and $\phi'$ are small. In a first approximation, we may neglect $\frac{L}{K}$, $\phi$, and $\phi'$ in the terms multiplied by L, and our equations of motion become

$$Mk^2\frac{d^2\phi}{dt^2} = L\cos npt - K(\phi-\phi'),\quad M'k'^2\frac{d^2\phi'}{dt^2} = K(\phi-\phi') \text{ ......... } (2),$$

which are linear equations to find $\phi$ and $\phi'$.—We are concerned to see in what cases $(\phi-\phi')$ tends to become large. From (2) we have

$$\frac{d^2(\phi-\phi')}{dt^2} + K\left(\frac{1}{Mk^2}+\frac{1}{M'k'^2}\right)(\phi-\phi') = \frac{L}{Mk^2}\cos npt \text{ ............ } (3).$$

The critical case is clearly when $n^2p^2 = K\left(\frac{1}{Mk^2}+\frac{1}{M'k'^2}\right)$ ............... (4);

and thus we have in the solution a term of the form $At\sin npt$, which would increase till the rod breaks.

The above equation simply expresses the fact that the period of variation in the couple $\frac{2\pi}{np}$ is the same as the time of vibration of the system when slightly disturbed from rest.

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**3659.** (Proposed by J. J. WALKER, M.A.)—Let D, E, F be the middle points of the sides BC, CA, AB respectively, of any spherical triangle, and let arcs AD, BE, CF meet in O; prove that $\frac{\tan AD}{\tan OD} = \frac{m+n}{m-n}$, where $m = 1+\cos a+\cos b+\cos c$, and $n = 1+\cos a$.

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I. *Quarternion Solution by* R. F. S.; *and* R. W. GENESE, B.A.

Let $\alpha, \beta, \gamma$ be the vectors from the centre to A, B, C. Then the vectors $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$, $\alpha+\beta+\gamma$ pass through D, E, F, O respectively; and by known formulæ

$$-T(\beta+\gamma)T\alpha\cos AD = S(\beta+\gamma)\alpha = S\alpha\beta+S\alpha\gamma = -\cos c-\cos b = -(m-n),$$

$$T(\beta+\gamma)T\alpha\sin AD = TV(\beta+\gamma)\alpha = \sin b+\sin c$$

$$-T(\beta+\gamma)T(\alpha+\beta+\gamma)\cos OD = S(\beta+\gamma)(\alpha+\beta+\gamma)$$
$$= -(2+2\cos a+\cos b+\cos c) = -(m+n),$$

$$T(\beta+\gamma)T(\alpha+\beta+\gamma)\sin OD = TV(\beta+\gamma)(\alpha+\beta+\gamma)$$
$$= TV(\beta+\gamma)\alpha = \sin b+\sin c;$$

therefore $$\frac{\tan AD}{\tan OD} = \frac{m+n}{m-n}.$$

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II. *Solution by the* PROPOSER.

Considering the sides of the triangle ABD cut by the transversal arc COF,

$$\frac{\sin AO}{\sin OD} = \frac{\sin AF}{\sin FB}\,\frac{\sin BC}{\sin CD} = 2\cos\tfrac{1}{2}a;$$

$$i.e.,\qquad \sin AD \cot OD - \cos AD = 2\cos\tfrac{1}{2}a;$$

$$\text{or,}\quad \tan AD \cot OD - 1 = \frac{2\cos\frac{1}{2}a}{\cos AD} = \frac{4\cos^2\frac{1}{2}a}{2\cos AD\cos\frac{1}{2}a} = \frac{2(1+\cos a)}{\cos b+\cos c};$$

$$\text{whence}\quad \frac{\tan AD}{\tan OD} = 1 + \frac{2(1+\cos a)}{\cos b+\cos c} = \frac{2(1+\cos a)+\cos b+\cos c}{\cos b+\cos c} = \frac{m+n}{m-n}.$$

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**3620.** (Proposed by the Rev. T. P. KIRKMAN, M.A., F.R.S.)—

A governess, famed for deportment,
And learned in every -ology,
Led daily, if not too wet,
A young and angelic assortment
Of loveliness blonde and brunette,
In number just eleven—
And this without any apology,
As Judy's leave was given—
To take their rounds
In Mr. Punch's pleasure grounds.

Soon won their budding charms
The Hero's loving eye:
One morning they descry
Him in their path, his arms
Loaded with glorious flowers;
And, while his beaming gaze
On all a gladsome blessing showers
To six he deals out six bouquets.

"Let not the ungifted sorrow,"
Said he, "six more to-morrow!
Or from this hand,
Or from that dainty stand,
Each on a silver tray,
Six gifts shall greet you day by day,
Or fruit, or flowers; and you shall say,
My sweet eleven,
Whether they be bouquets or bunches,
That under heaven
There are no gifts like Mr. Punch's.
First come first served;
But let this law
Be well observed,
That never a five of you shall draw
Their gifts together on any morn
Who all at once before have borne
Five gifts away:
So wisely fix
The expectant six,
For every day:
If five have been five in a six before,
The trays will adorn the stand no more.

From her gossamer chaise
In the summer heaven,
Titania, queen of the pixies,
The promise overheard,
And vowed, for days
Six times eleven,
To help them choose the sixes:
The Fairy kept her word.

What lady has skill
The lists to fill?
Who does it, I'll love her,
And handsomely glove her.

*Solution by* N'IMPORTE.

It is clear that no five will be a second time at once decorated, so long as no four are for a second time at once undecorated. The problem is, therefore, simply a reproduction of Mr. LEA's Question 2244, to make 66 5-plets out of 11 elements, so that no two of them shall have a common 4-plet. The sixes required are the 6-plets complementary to the 5-plets given by Mr. LEA in his Solution on pp. 35, 36 of Vol. IX. of the *Reprint*.

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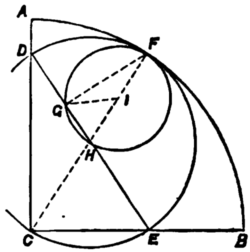
**3673.** (Proposed by the Rev. J. WHITE.)—Prove by a geometrical method, that a ladder slipping down a wall from a vertical to a horizontal position, continually touches a hypocycloid generated in a circle with radius equal to the length of the ladder, by a rolling circle of radius one-fourth that length.

*Solution by the* PROPOSER; *and others.*




Let AC represent the ladder in its vertical, and BC in its horizontal position; and draw the circle AFB.

The ladder will always be the diameter of a circle whose radius is half the length of the radius of the circle AFB, and which rolls down the quadrant AFB.

For, take any position of the circle CDFE, DE is a diameter of the circle (for the angle at C is a right angle) and therefore is a position of the ladder.

Further, the arc DF subtends the angle DHF at H, the centre of the circle CDFE, which is double the angle ACF, but the arc AF in the circle whose radius is twice that of the circle CDFE, subtends this angle; therefore the arc AF is equal to the arc DF, and the points D and A must meet as the circle rolls on the fixed circle AFB.

Let a circle whose radius is one-fourth AC commence to roll simultaneously with the circle CDFE, and it will occupy the position FGH. The centres C, H, I of the three circles, and the point of contact, will be in one straight line.

Draw FG to the point where DE cuts the least circle.

The arc FG subtends the angle GIF which is double the angle DHF, which the arc DF subtends at the centre of a circle of double the radius; therefore the arc FG = arc FD = arc FA.

Therefore the point G, where DE cuts the lesser rolling circle, is a point on the curve generated by it starting from A.

Further, the line FG must be always perpendicular to the diameter DE of the circle CDFE; and therefore the latter must be tangent to the curve generated by the point G, for FG is normal to the curve, since it is generated by its motion round the point F.

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**3702.** (Proposed by R. W. GENESE, B.A.)—Prove geometrically the known theorem that if forces be represented in magnitude and direction by the perpendiculars drawn from any point O within a regular polygon on its $n$ sides, the resultant will pass through the centre C of the polygon and be represented by $\frac{1}{2}n \,.\, OC$.

*Solution by* Professor TOWNSEND, M.A., F.R.S.

Denoting by A, B, C, &c., the several sides of the polygon; by PA, PB, PC, &c., the several perpendiculars to them from any internal point P; by OA, OB, OC, &c., the several from the centre O; and by OX, OY, OZ, &c., the several from the same point O to the system of lines PA, PB, PC, &c. Then, since PA = PX + OA, PB = PY + OB, PC = PZ + OC, &c., and since the system of forces OA, OB, OC, &c., evidently make equilibrium, therefore the resultant of the system PA, PB, PC, &c., coincides in direction and magnitude with that of the system PX, PY, PZ, &c.;

but, the system of points X, Y, Z, &c., forming evidently, according as $n$ is odd or even, the $n$ vertices of one regular polygon, or the $2(\frac{1}{2}n)$ vertices of two coincident regular polygons, inscribed to the circle on OP as diameter, the resultant of the latter system is in magnitude and direction $= n(\frac{1}{2}\text{PO})$, and therefore, &c.

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**3598.** (Proposed by Judge SCOTT.)—To find three integral square numbers such that the sum of any two of them shall be a square number.

I. *Solution by the* PROPOSER.

Let $x^2$, $y^2$, $z^2$ be the numbers; then

$$x^2+y^2=\square \;......(1),\quad x^2+z^2=\square\;......(2),\quad y^2+z^2=\square\;......(3).$$

Put $x^2+y^2=(y-m)^2$, and $x^2+z^2=(z-n)^2$; then

$$y=\frac{m^2-x^2}{2m},\ \text{and}\ z=\frac{n^2-x^2}{2n}.$$

Substituting in (3), putting $m^2+n^2=s^2$, and reducing, we have

$$s^2x^4-4m^2n^2x^2+m^2n^2s^2=\square \quad ........................\ (4).$$

Now put $(4)=s^2x^4$, and we have $x=\frac{1}{2}s$.

Let $$m=\left(\frac{p^2-q^2}{p^2+q^2}\right)s,\ \text{and}\ n=\left(\frac{2pq}{p^2+q^2}\right)s;$$

then, taking $s=16pq(p^4-q^4)$, we obtain, finally,

$$x=8pq(p^4-q^4),\quad y=2pq(3p^2-q^2)(p^2-3q^2),$$
$$z=(p^2-q^2)(4pq+p^2+q^2)(4pq-p^2-q^2).$$

Take $p=2$, $q=1$; then $x=240$, $y=44$, $z=117$, which are probably the roots of the least numbers which will satisfy the conditions.

II. *Solution by* Professor GILL.

Put $$x^2+y^2=a^2,\quad y^2+z^2=b^2,\quad x^2+z^2=c^2 ...........\ (1, 2, 3);$$

then we have $$a^2+z^2=b^2+x^2=c^2+y^2\ ...........................\ (4).$$

Assume $$b=a\cos A+z\sin A,\quad x=z\cos A-a\sin A,$$
$$c=a\cos B+z\sin B,\quad y=z\cos B-a\sin B;$$

then (4) is satisfied, and (1) becomes

$$(\cos^2 A+\cos^2 B)z^2-a(\sin 2A+\sin 2B)z=a^2(1-\sin^2 A-\sin^2 B)...\ (5).$$

Now take $A+B=90°$, and (5) gives $z=2a\sin 2A$; and by substitution

$$x=a\cos A(4\sin^2 A-1),\quad y=a\sin A(4\cos^2 A-1).$$

Take $\cot\frac{1}{2}A=\frac{m}{n}$, so that we shall have

$$\sin A=\frac{2mn}{m^2+n^2},\quad \cos A=\frac{m^2-n^2}{m^2+n^2};$$

then, if we put $a = (m^2 + n^2)^3$, we have, finally,

$$x = 2mn(m^2-3n^2)(3m^2-n^2), \quad y = n^2(3m^2-n^2)^2 - m^2(n^2-3m^2)^2,$$
$$z = 8mn(m^4-n^4).$$

Let $m=2$, and $n=1$; then $x=44$, $y=117$, we have $z=240$.

[This elegant method of solving such problems was first published in 1848, by Professor CHARLES GILL of New York, in his *Application of the Angular Analysis to the Solution of Indeterminate Problems of the Second Degree.* As Prof. GILL's book is out of print and extremely difficult to obtain, we have thought it desirable to reproduce one or two examples of his method.]

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III. *Solution by* A. MARTIN.

Put $\quad x^2+y^2 = u^2, \quad y^2+z^2 = v^2, \quad$ and $\quad x^2+z^2 = w^2$ ......... (1, 2, 3).

By transposition we have $\quad x^2 = u^2-y^2 = w^2-z^2$ ........................... (4).

To satisfy $u^2-y^2 = w^2-z^2$, assume $u = a(r^2-s^2)$, $y = b(r^2-s^2)$,

$$w = a(r^2+s^2)-2brs, \quad \text{and} \quad z = b(r^2+s^2)-2ars.$$

If, now, $a = p^2+q^2$ and $b = 2pq$, $u^2-y^2$ and $w^2-z^2$ will both be squares.

Hence, if we take $\quad x = (p^2-q^2)(r^2-s^2), \quad y = 2pq(r^2-s^2)$,

and $\quad z = 2pq(r^2+s^2)-2rs(p^2+q^2)$,

(1) and (3) will be satisfied.

Substituting in (2), dividing by $4r^2s^2$, arranging the terms, and putting $m = \dfrac{r^2+s^2}{rs}$, we have

$$p^4-2mqp^3+2(m^2-1)q^2p^2-2mq^3p+q^4 = \square \quad \text{............... (5).}$$

Put (5) $= (p^2-mqp-q^2)^2$, and by involution and reduction we find

$$\frac{p}{q} = \frac{4}{m} = \frac{4rs}{r^2+s^2}.$$

Taking $p = 4rs$ and $q = r^2+s^2$, we have

$$x = (r^2-s^2)(14r^2s^2-r^4-s^4), \quad y = 8rs(r^4-s^4), \quad z = 2rs(3r^4-10r^2s^2+3s^4).$$

If we put $r=2$ and $s=1$, we have $x=117$, $y=240$, $z=44$.

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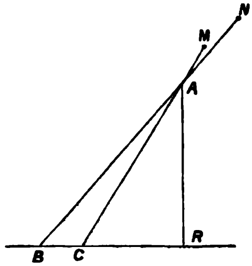
**3703.** (Proposed by T. MITCHESON, B.A.)—A pole, of length $l$, rests with one end on a horizontal plane, the other projecting over a perpendicular wall, and makes an angle $\alpha$ with the plane; the hypotenuse of the right-angled triangle thus formed is $\frac{p}{q}l$; the pole partially breaks, so that the portion between the foot and the point where it breaks lies on the plane, its length being $\frac{m}{n}l$; and the part *now* projecting, $\frac{mq}{np}l$, breaks off; show that

$$p(n^2p^2+m^2q^2-2mnpq\cos\alpha)^{\frac{1}{2}} - q\left\{np-m(p+q)\right\} = 0.$$

*Solution by* Q. W. King; J. H. Slade; *the* Proposer; *and others.*




Let BR be the plane; AR the wall; and BAN, BCAM the pole in its first and second positions; then $AB = \frac{p}{q}l$, $\angle ABR = \alpha$,

$$BC = \frac{m}{n}l, \text{ and } AM = \frac{mq}{np}l.$$

Now $$l = BC + CA + AM$$
$$= \frac{m}{n}l + CA + \frac{mq}{np}l;$$

therefore $$CA = \frac{l}{np}\left\{np - m(p+q)\right\};$$

also $$AC = (AB^2 + BC^2 - 2AB.BC\cos ABC)^{\frac{1}{2}}$$
$$= \frac{l}{nq}(n^2p^2 + m^2q^2 - 2mnpq\cos\alpha)^{\frac{1}{2}}.$$

Equating these two values of AC, we get the relation required.

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**3739.** (Proposed by A. B. Evans, M.A.)—Find two positive cube numbers, besides 1 and 8, such that their sum shall be 9.

*Solution by* Samuel Bills.

Let $x+y$ and $x-y$ be the roots of the two required cubes, then we must have $(x+y)^3 + (x-y)^3 = 9$, or $2x^3 + 6xy^2 = 9$; whence $y^2 = \frac{9-2x^3}{6x}$; we must therefore have $6x(9-2x^5) = \square$. Put $x = 3w$, then we must have $2w(1-6w^3) = \square$. This will be a square when $w = \frac{1}{2}$; assume therefore $w = \frac{1}{2} + v$, then we shall have to find

$$\tfrac{1}{4} - 4v - 18v^2 - 24v^3 - 12v^4 = \square = (\tfrac{1}{2} - 4v - 34v^2)^2 \text{ (suppose)},$$

from which we find $v = -\frac{37}{146}$, and $w = \frac{18}{73}$; whence $x = \frac{54}{73}$. From this we find $y = \frac{595}{438}$. This gives for the two numbers $x+y = \frac{919}{438}$, and $x-y = -\frac{271}{438}$, one of which is *negative*, and therefore inadmissible.

Now, in the identity $$a^3 + b^3 = \left(a.\frac{a^3+2b^3}{a^3-b^3}\right)^3 + \left(b.\frac{b^3+2a^3}{b^3-a^3}\right)^3,$$

substitute $\frac{919}{438}$ for $a$, and $-\frac{271}{438}$ for $b$; and we shall find

$$\left(\frac{919}{438}\right)^3 - \left(\frac{271}{438}\right)^3 = \left(\frac{676702467503}{348671682660}\right)^3 + \left(\frac{415280564497}{348671682660}\right)^3 = 9.$$

It will be seen that the latter are *positive* numbers.

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**3674.** (Proposed by T. Cotterill, M.A.)—1. In a plane, if the perpendiculars on a line from the points $a, b, c \ldots l, n, p$ be denoted by the cor-

responding letters, whilst $(abc)$ denotes the area of the triangle formed by the corresponding points, then the value of the expression

$$\frac{1}{p}\left(\frac{(pab)}{a.b}+\frac{(pbc)}{b.c}+\ldots+\frac{(pln)}{l.n}+\frac{(pna)}{n.a}\right)$$

is independent of the position of the point $p$. Thus it may be any one of the fixed points, in which case two of the terms disappear.

2. By changing the meaning of the symbols, the expression is its own dual and true for the sphere.

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*Solution by* R. W. GENESE, B.A.

Let the fixed line be the axis of $x$, and let $(x_1, y_1) \ldots (x_2, y_2)$ be the coordinates of $abc\ldots$, $(x, y)$ those of any point $p$; then we have

$$(pab) = x(y_1-y_2)-y(x_1-x_2)+x_1y_2-x_2y_1;$$

therefore $$\frac{(pab)}{a.b} = x\left(\frac{1}{y_2}-\frac{1}{y_1}\right)-y\left(\frac{x_1-x_2}{y_1y_2}\right)+\frac{x_1}{y_1}-\frac{x_2}{y_2}.$$

Hence $$\frac{(pab)}{a.b}+\frac{(pbc)}{b.c}+\ldots+\frac{pna}{n.a} = -y\left\{\frac{x_1-x_2}{y_1y_2}+\ldots\right\} \propto y \text{ or } p;$$

therefore $$\frac{1}{p}\left\{\frac{(pab)}{a.b}+\frac{(pbc)}{b.c}+\ldots\right\} \text{ is independent of } p.$$

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**3562.** (Proposed by the Rev. T. P. KIRKMAN, M.A., F.R.S.)—

Titania, from her chariot, drawn
By gorgeous dragon-flies,
Beheld the game on Punch's lawn,
And conned an arch surprise
For Oberon. "He 's mighty wise,
To quiz my fashions in Fairyland;
And grown so dreadfully rude, he
Derided my last new ballet in air:
I'll make him aware
That I have him in hand,
By a dance and a puzzle from Judy."
And she wiled from Judy, asleep in her chair,
The secret I would have you share.

First, to her myriad sylphs and gnomes
She taught the lesson in their homes
Of forest gloom and cave;
And all their power
At moonlit hour
Came spinning in circles o'er land and wave.

Then she, in potent witchery
Of smiles and queenly pomp arrayed,
Began, "My lord, will you ride with me?"
King Oberon obeyed.
She lashed her lightning team, and soon
They stood on the chin of the crescent moon.

O what a chorus of curve and spangle,
In every plane, at every angle,
Burst on the cloudless night!
A sylph or a gnome for every gem
In midnight's ebon diadem,
In number $R$, the mighty sum
Of stars and hosts of Elfendom!
From sea to zenith, in foams of light,
They surge in vast and tiny rings,
And each on each can read from far
The flashing name of the other's star,
Jewelled upon his rainbow wings.

Once now she winds her fairy horn;
And all, from Cancer to Capricorn,
Is still, and every sprite
On the star attent,
In the firmament,
Of him upon his right.

Then, rising in her pearly car,
And snatching Oberon's wand, she drew
Swift random lines from star to star,
Twelve in a minute, and every two
Shot to each other a lightning call,
A silver braid on midnight's pall;
Choosing the pairs with random gaze
O'er all the twinkling floor,
But no one twice, the wand she plays
While chance or sportive fancy says,

One couple more;
And, twelve in a minute, flashes blaze
A second—and 'tis o'er.

Each said, "M. N., whoever clasps
The left of either lets go it, and grasps
The left of the other." Your fancy's play
Will fill up the picture,
Unless you restrict your
Taste to analysis,
So that paralysis
Eats the pith of your life away.

Picture the seething whirl of elves,
As, at every signal, a circle brake
And flashed again twain—or, long as the Snake
That girds the pole,
Two broken ones rush from East and West
Within five seconds to join themselves—
As the welkin rings with the laugh of the rest—
A glittering circled whole.
The flashes ceased, and the rings went on
Dancing unbroken. "Encore, encore!"
Said Oberon:
Then played the magic wand the more,
At random separate pairs, uncaring
Of what was touched or not before
This second pairing;
And summoned sylph and gnome go tearing
O'er moonlit sea and shore,
With all to the prisoned hand that cling,
To form the flashing ring.

Again she tried;
Again desired,
Beginning as before anew,
Each random sweep through other two,
She spurred the host of circles bright
For Oberon's delight

Then, as she sounds her silver horn,
The fairy pageant fades,
Melting in moonbeams, ere the morn
The realm of night invades.

"Ha, Ha!" quoth he,
"'Twould something be,
If, at your second pause,
You so had known the laws
Of changing pairs, as to recall
The circles which began the ball!"

"O king of elves, you're wonder-wise,"
Said she, with vengeful eyes;
"I challenge your Royal Elegance
To tell, in terms of *R*, the chance
There was, when I began,
That all the rings, that dying span,
In number and order of names should play
As in the first array."

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*Solution by the* PROPOSER.

The circles formed by the R Sylphs and Gnomes at their first appearance, were those of a certain substitution S, no matter of what order, in which Titania was required by Oberon's hint so to operate by three substitutions $\nu$, $\mu$, $\theta$, each of the second order, as to produce the result $\theta\mu\nu S = S$, or $\theta\mu\nu = 1$, a triplet of permutable pairs, $\mu\nu = \nu\mu = \theta$, &c. This places before us the problem that arises when the restriction $A > 2$ is removed from Quest. 3402. We have to determine in how many ways $\theta$ of the second order, made with R elements, can be broken into the product of $\mu$ and $\nu$, both of the second order. The number of groups of two, $1\theta$, made with R elements on the partition $R = 2(a+1)+b$, $(A=2, B=1$, p. 75 of Vol. XVI. of the *Reprint*, line 9) is

$$U = \Sigma_a W = \Sigma \left\{\Pi R : [\Pi(a+1)\Pi b \,.\, 2^{a+1}]\right\}; \quad (a \geqq 0),$$

of which each gives a different group $1\theta$. If, withdrawing the restriction $A > 2$ in Quest. 3402, we proceed by the method of that page 75 to complete $1\theta$ into the group of four $1\theta\mu\nu$, where $\mu\nu = \nu\mu = \theta$, we shall obtain $\mu\nu$, as there indicated, taking all positive and zero values of $\alpha$ and $\beta$, in

$$P = \Sigma_{\alpha\beta} \frac{2^{a+1-\alpha}\,\Pi . a+1)\Pi b}{2^{\alpha+\beta}\,\Pi(a+1-2\alpha)\,\Pi(b-2\beta)\,\Pi_\alpha \Pi_\beta}$$

ways, comprising both $1\theta\mu\nu$ and $1\theta\nu\mu$; and if we operate on the groups $1\mu$ and $1\nu$ in like manner, we shall complete among our results $1\mu\theta\nu$, $1\mu\nu\theta$, $1\nu\theta\mu$, and $1\nu\mu\theta$; that is, the entire number UP of results will contain six times every triplet of permutable pairs $\theta\mu\nu = \mu\nu\theta$, &c., in as many arrangements. Wherefore, if no error lay in the removal of the restriction $A > 2$, $\frac{1}{6}$UP would be the exact number of triplets required, and UP the number

of ways in which Oberon's unreasonable fancy could have been gratified. But there is an error in such removal, which is this: that when $\alpha=\beta=0$, *i.e.*, when after forming $\mu$ and $\nu$ by the rule, we transpose no pair of the $a+1$ circles of two, and no pair of the $b$ undisturbed elements, we merely complete the systems $1\theta 1\theta$ and $1\theta\theta 1$; but if either $\alpha$ or $\beta>0$, we complete correctly a system $1\theta\mu\nu$. Thus we can correct the only possible error, whatever be our group of two, $1\theta$, by writing $P-2.0^{\alpha+\beta}$ instead of P, and the exact number of different triplets of permutables possible is $\frac{1}{6}U(P-2.0^{\alpha+\beta})=K$; or

$$6K=\Pi R.\Sigma_{a,\alpha,\beta}\left\{\frac{1}{2^{2\alpha+\beta}\Pi(a+1-2\alpha)\Pi(b-2\beta)\Pi_\alpha\Pi_\beta}-\frac{0^{\alpha+b}}{2^a\Pi(a+1)\Pi b}\right\},$$

taken for all values positive or zero of $a$, $\alpha$, $\beta$, is the number of ways in which Titania could have made her vanishing circles identical with those which began the ball. Hence the chance which Oberon was challenged to find, since U is the number of all substitutions of the second order, is

$$\frac{6K}{U(U-1)(U-2)}.$$

The theorem proved in Quest. 3402 may be thus stated without the restriction $A>2$. Let $N=aA+bB+\ldots$ be any partition of N, and $\Theta$ any substitution formed on it, A, B, ... being in descending order. The number of ways in which $\Theta=\theta\theta'$ can be satisfied, $\theta$ and $\theta'$ being of the second order, is $\Sigma_{\alpha\beta\gamma\ldots}\left\{\frac{A^{a-\alpha}B^{b-\beta}\ldots\Pi a.\Pi b\ldots}{\Pi(a-2\alpha)\Pi(b-2\beta)\ldots\Pi\alpha.\Pi b\ldots}-2.0^{A+\alpha+\beta-2}\right\}$,

where $\alpha$, $\beta$, $\gamma$ are any zero or positive numbers.

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**3625.** (Proposed by Professor CROFTON, F.R.S.)—A vertical rod, unstrained, of trifling weight, is immovably fixed at two points; a heavy weight is clamped on it at a given point between them: find the pull produced on the upper portion and the thrust on the lower.

Also, taking the weight of the rod into account, find the stresses on the two immovable supports, and the tension or thrust at all points of the rod.

Is it a valid solution to the first part of the question, to take the rod first slightly inclined to the vertical, to find thus the stresses on the two supports, and infer that the result holds in the limit when the rod becomes vertical?

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*Solution by* G. S. CARR.

Let A, B be the extremities of the rod and also the points of support; C an intermediate point where the weight W is attached; P, Q the strains at A and B; $e$ the modulus of elasticity of the rod for extension; $e'$ the modulus for compression; $s$ = area of section of rod; $AB=a$; $AC=b$; $z$ = the small distance through which the point C descends when W is attached. Then, premising that all distances measured along the rod are to be understood as taken when the fibres are in their normal state, we shall have, in the first case, neglecting the weight of the rod,

$$\frac{P}{se} = \frac{z}{b}, \quad \frac{Q}{se'} = \frac{z}{a-b}, \quad P + Q = W;$$

therefore $$P = \frac{e(a-b)}{e(a-b)+e'b}W, \quad Q = \frac{e'b}{e(a-b)+e'b}W.$$

In the next place, suppose the rod uniformly heavy; and first let no weight be attached to it. There will now be a neutral point D above which the fibres of the rod will be extended, and below which they will be compressed.

The tension at a point E above D = the weight of ED = $(h-x)w$, where $h$ = AD, $x$ = AE, and $w$ = the weight of a unit of the rod's length.

Similarly the pressure at a point E below D = the weight of DE = $(x-h)w$; and to find $h$, we must put the extension of AD = the compression of DB.

Now the extension of an element $dx$ in AD $= \frac{(h-x)w}{se}dx$; therefore the whole increase in the length of AD, and decrease in DB, are

$$\int_0^h \frac{(h-x)w}{se}dx = \frac{h^2 w}{2se}, \quad \int_h^a \frac{(x-h)w}{se'}dx = \frac{(a-h)^2 w}{2se'}.$$

Equating the two results we find $h = \frac{e^{\frac{1}{2}} + e'^{\frac{1}{2}}}{ae^{\frac{1}{2}}}$.

Now let W = $kw$, be a weight attached as before at C. The effect will be to make the neutral point D approach to C.

First when C is below D,

the tension at a point in AD $= (h-x)w$,
the pressure at a point in DC $= (x-h)w$,
and at a point in CB $= (x-h+k)w$.

Thus the strain at any point is known when $h$ is known; and $h$ is found as before by equating the extension of AD to the compression of DB, that is

$$\int_0^h \frac{(h-x)w}{se}dx = \int_h^b \frac{(x-h)w}{se'}dx + \int_b^a \frac{(x-h+k)w}{se'}dx,$$

therefore $$h = \frac{ae - \left\{e\left[a^2e' - 2k(a-b)(e-e')\right]\right\}^{\frac{1}{2}}}{e-e'}.$$

If C be above D, we find $h = \frac{ae - \left\{e'\left[a^2e + 2kb(e-e')\right]\right\}^{\frac{1}{2}}}{e-e'}$.

The value of W which will make C the neutral point is found by putting $h = b$ in the quadratic referred to, which gives $k = \frac{b^2e' - e(a-b)^2}{2e(a-b)}$.

For any greater value of $k$ the whole of AC will be in a state of tension, and the whole of CB will be compressed. Part of W will then add to the tension of AC, and the remaining part to the compression of CB. Let the two parts be $pw$ and $qw$, so that $p + q = k$,

the tension at a point in AC $= (b-x+p)w$,
the pressure at a point in BC $= (x-b+q)w$,

and $p$ and $q$ are determined as before from the equation of equilibrium

$$\int_0^b \frac{(b+p-x)w}{se}dx = \int_b^a \frac{(x+q-b)w}{se'}dx,$$

therefore $p = \dfrac{e(a-b)^2+2ke(a-b)-b^2e'}{2be'+2e(a-b)}, \quad q = \dfrac{2kbe'+b^2e'-e(a-b)^2}{2be'+2e(a-b)}.$

It would not be a valid solution of the first case to take the rod slightly inclined to the vertical, and to resolve the weight merely into two vertical pressures at A and B. It would be necessary to resolve W along and perpendicular to the rod. The former component produces strains at A and B in the manner already shown, and the latter vanishes when the rod becomes vertical, and the results are the same as before.

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**3653.** (Proposed by Professor WOLSTENHOLME, M.A.)—A heavy particle moves in a circular groove, radius $a$, in a vertical plane, the resistance of the air being $\frac{1}{2}c^{-1}$ (vel.)$^2$: prove that if it start from a point in the upper half at which the tangent makes an angle $\tan^{-1}(ca^{-1})$ with the horizon, it will just describe a semicircle before turning, and during this motion its kinetic energy will vary as the distance from the diameter bounding the semicircle.

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I. *Solution by* J. J. WALKER, M.A.

Let $\theta$ be the angle which the radius through the position of the particle at any instant makes with the horizontal radius, on the side from which the particle starts; $s$ the arc described, measured downwards from the same radius. Then we have

$$v\frac{dv}{ds} = -\frac{v^2}{2c} + g\cos\theta = -\frac{v^2}{2c} + g\cos\frac{s}{a}, \quad \text{or} \quad \frac{1}{2}\left(\frac{v^2}{c} + \frac{2v\,dv}{ds}\right) = g\cos\frac{s}{a}.$$

Multiplying both sides by $e^{\frac{s}{c}}ds$, and integrating, we have

$$\tfrac{1}{2}e^{\frac{s}{c}}v^2 = \frac{gace^{\frac{s}{c}}}{a^2+c^2}\left(a\cos\frac{s}{a} + c\sin\frac{s}{a}\right) + k.$$

Writing $\dfrac{a}{c} = \tan\alpha$, we have $\tfrac{1}{2}v^2e^{\frac{s}{c}} = ga\cos\alpha e^{\frac{s}{c}}\sin(\alpha+\theta) + k.$

Initially $v=0$, $\theta=-\alpha$, consequently $k=0$. When $\theta+\alpha=\pi$, that is, when the particle has described a semicircle from its initial position, $v$ again becomes zero, and the particle begins to return. Returning to the general value of $v$, since $k=0$, $\frac{1}{2}mv^2 = gma\cos\alpha\sin(\alpha+\theta)$, or $\propto a\sin(\alpha+\theta)$, which is its perpendicular distance from the diameter through its initial position.

The reaction of the groove at each position of the particle (the weight of which $=w$) is equal to $w\left\{(1+2\cos^2\alpha)\sin\theta + \sin 2\alpha\cos\theta\right\}$,

and changes from outwards to inwards when $\tan\theta = -\dfrac{\sin 2\alpha}{1+2\cos^2\alpha}$.

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II. *Solution by* Professor TOWNSEND, M.A., F.R.S.

Denoting by $\theta$ the angle, measured in the direction of the motion, between

the radius of the circle pointing vertically upwards and that revolving with the particle, the equation of the angular motion, during the entire interval between starting and turning, is evidently

$$\frac{d^2\theta}{dt^2} = \frac{g}{a}\sin\theta - \frac{a}{2c}\frac{d\theta^2}{dt^2} \qquad \text{.......................... (1);}$$

and, to prove both parts of the question, it is manifestly necessary only to show that, under the initial circumstances supposed, its first integral assumes the simple form $\qquad \frac{d\theta^2}{dt^2} = 2k\sin(\theta - \alpha)$ .............................. (2),

where $\alpha$ is the prescribed initial value of $\theta$, and $k$ a constant to be determined.

Since from (2), assumed as first integral of (1), by differentiation with respect to $t$, $\qquad \frac{d^2\theta}{dt^2} = k\cos(\theta-\alpha)$ .............................. (3);

therefore, substituting from (2) and (3) in (1), and equating to 0 the coefficients of $\cos\theta$ and of $\sin\theta$ in the result, we have, to determine $\alpha$ and $k$, the two equations

$$\cos\alpha - \frac{a}{c}\sin\alpha = 0, \quad k\left(\sin\alpha + \frac{a}{c}\cos\alpha\right) - g\frac{c}{a} = 0 \qquad \text{......... (4),}$$

from which, since at once $\tan\alpha = \frac{c}{a}$, and $k = \frac{g\tan\alpha}{(a^2+c^2)^{\frac{1}{2}}}$, therefore, &c.

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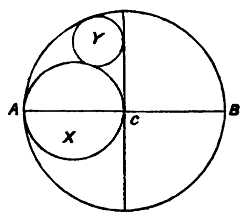
**3601.** (Proposed by the Rev. W. H. LAVERTY, M.A.)—ACB is the diameter of a circle, centre C; P is a parabola, focus C, vertex A; H is a hyperbola with one asymptote parallel to ACB, with one focus at C, and touching P and the circle. Show that the transverse axis of H is equal to the radius of the circle.

*Solution by the* PROPOSER.




Reciprocate with respect to the given circle. Then P becomes a circle X described on AC as diameter, H becomes a circle Y, touching X and the original circle, and also the perpendicular radius. Draw the radius CMN, MN being a diameter of Y.

Now it is easily proved that

$MN = \frac{1}{3}CA = \frac{1}{3}CN$; therefore $CM = \frac{2}{3}CN$; therefore, if $m$ be the vertex of the hyperbola corresponding to the tangent at M (N being the other vertex), $Cm = 2CN$; $\therefore\ mN = CN$.

**3568.** (Proposed by Professor TOWNSEND.)—A mass of homogeneous matter is formed into the solid of maximum attraction, for the law of nature, of a particle placed on its surface. Show that the ratio of its maximum attraction to that of a sphere of the same mass and material on the same particle placed on its surface $= 3 : \sqrt[3]{(25)}$.

*Solution by the* PROPOSER.

Denoting by M, A, $\rho$, $a$ the mass, maximum attraction, density, and axis of the solid, which, on the simplest elementary principles, is seen at once to be that bounded by the surface of revolution the polar equation of whose meridian is $r^2 = a^2 \cos\theta$, and M′, A′, $\rho'$, $a'$ the same for any homogeneous sphere; then since, by direct integration or otherwise, $M = \frac{4}{15}\pi\rho a^3$, $A = \frac{4}{5}\pi\mu\rho a = \frac{3\mu M}{a^2}$, and $M' = \frac{1}{6}\pi\rho' a'^3$, $A' = \frac{2}{3}\pi\mu\rho' a' = \frac{4\mu M'}{a'^2}$, and since, by hypothesis, $M = M'$, and $\rho = \rho'$; therefore at once $a^3 : a'^3 = 5 : 8$, and $A : A' = 6a : 5a'$, and therefore, &c.

**3721.** (Proposed by J. HOPKINSON, D.Sc., B.A.)—A string AB is stretched between two points A and B. A is fixed, but B moves according to the law of displacement $A \sin nt$ perpendicular to the string, A being small. At the beginning of the time the string is straight and at rest, B being in its mean position and moving with velocity $nA$. Prove that the motion of the string will be made up of a number of simple vibrations of periods the same as the times of vibration of the string when fixed at A and B, and a vibration whose period is the same as that of the forced vibration at B, viz. $\frac{2\pi}{n}$. Determine the motion completely.

*Solution by the* PROPOSER.

The equation of motion of the string may be written

$$\frac{d^2\xi}{dt^2} = a^2 \frac{d^2\xi}{dx^2} \quad\text{.............................} (1),$$

$a$ being the velocity of transmission of a wave along the string, and $x$ being measured from the fixed point A. We have also the conditions

$\xi = 0$ when $x = 0$ for all values of $t$, $\xi = A \sin nt$ when $x = l$ ... (2, 3);

$\xi$ and $\frac{d\xi}{dt}$ vanish for all values of $x$ from 0 to $l$ when $t = 0$ ...... (4, 5).

The general solution of (1) is $\xi = f(at - x) + F(at + x)$.................. (6).

Putting $x = 0$, we have, by (2), $0 = f(at) + F(at)$,

$\therefore\ \xi = f(at - x) - f(at + x)$; and by (3), $A \sin nt = f(at - l) - f(at + l)$;

whence
$$\xi = \frac{A}{\sin\frac{nl}{a}} \sin nt \sin\frac{nx}{a} + \Sigma C_p \sin\frac{p\pi at}{l} \sin\frac{p\pi x}{l},$$

which shows that the motion is made up of simple vibrations, one of which has the same period as the disturbance at B, the others the natural periods of vibration of the string fixed at both ends. It should be noticed that if

$nl$ be a multiple of $\pi a$, *i.e.* if the period of the disturbance be the same as any one of the times of the string, then, no matter how small the disturbance may be at B, the disturbance produced in the string will be so great as to render equation (1) quite inapplicable.

Applying (5), we must make

$$\frac{nA \sin \frac{nx}{a}}{\sin \frac{pl}{a}} + \Sigma C_p \frac{p\pi a}{l} \sin \frac{p\pi x}{l} = 0,$$

whence $\quad C_p \dfrac{p\pi a}{l} \dfrac{l}{2} = \dfrac{nA}{2 \sin \frac{nl}{a}} \displaystyle\int_0^l \left\{ \cos\left(\frac{n}{a} - \frac{p\pi}{l}\right) x - \cos\left(\frac{n}{a} + \frac{p\pi}{l}\right) \right\} x,$

$$C_p = \frac{nA}{p\pi a \sin \frac{nl}{a}} \left\{ \frac{\sin\left(\frac{nl}{a} - p\pi\right)}{\frac{n}{a} - \frac{p\pi}{l}} - \frac{\sin\left(\frac{nl}{a} + p\pi\right)}{\frac{n}{a} + \frac{p\pi}{l}} \right\} = (-1)^p \frac{2nalA}{n^2l^2 - p^2\pi^2a^2},$$

which determines the motion.

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**3754.** (Proposed by C. TAYLOR, M.A.)—From a point P on the circumscribed circle of a triangle, whose orthocentre is O, perpendiculars PQ, PR are drawn to two of the sides. Show that OP, QR intersect on the nine-point circle.

*Solution by* R. F. S.

QR is the pedal line of P. We know that OP is bisected by QR in the point where it meets it. But every line from the orthocentre to the circumscribed circle is bisected by the nine-point circle. Whence the theorem follows.

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**3550.** (Proposed by the EDITOR.)—Show that if we eliminate $\xi$ and $v$ from $U \equiv \xi^2 + v^2 - \left\{e\xi + h(\xi^2 + v^2)\right\}^2 = 0, \quad \dfrac{dU}{xd\xi} = \dfrac{dU}{ydv} = \dfrac{\xi dU}{d\xi} + \dfrac{vdU}{dv}$;

or $\phi$ from $\quad \dfrac{x}{\cos\phi - e\cos 2\phi} = \dfrac{y}{\sin\phi - e\sin 2\phi} = \dfrac{h}{(1 - e\cos\phi)^2}$;

or $\psi$ from $\quad \dfrac{x}{2e + \cos\psi - e\cos^2\psi} - \dfrac{y}{u\sin\psi\,(1 - 2e^2 - e\cos\psi)} = \dfrac{h}{u^2}$,

where $u^2 = 1 - e^2$; the resulting equation in $x$ and $y$ is

$$u^2(x^2 + u^2y^2)^2 - 2(5 - 4e^2)\,ehx^3 - 2(5 + 4e^2)\,eu^2hxy^2$$
$$-(1 - 32e^2 + 16e^4)\,h^2x^2 - (1 - 20e^2 - 8e^4)\,h^2y^2 + 8(1 - 4e^2)\,eh^3x - 16e^2h^4 = 0.$$

*Solution by the* PROPOSER; *and* G. S. CARR.

The elimination is the one required for finding the equation of the first

negative focal pedal of an ellipse (see *Reprint*, Vol. XVI., pp. 77—83, Solutions of Question 3430).

Put $\xi = \frac{\cos\phi}{p}$, and $v = \frac{\sin\phi}{p}$; then we have

$$\xi^2+v^2 = \frac{1}{p^2}, \text{ and } e\xi+h(\xi^2+v^2) = \frac{ep\cos\phi+h}{p^2}.$$

But $\xi^2+v^2 = \left\{e\xi+h(\xi^2+v^2)\right\}^2$ from the equation $U=0$;

therefore $$\frac{ep\cos\phi+h}{p^2} = \frac{1}{p}, \text{ and } ep\cos\phi+h = p \text{ .................. (1).}$$

Differentiating $U=0$, and substituting the above values in the results, we obtain $\frac{1}{2}\cdot\frac{dU}{d\xi} = \frac{\cos\phi}{p} - \frac{1}{p}\left(e+\frac{2h\cos\phi}{p}\right)$, $\frac{1}{2}\cdot\frac{dU}{dv} = \frac{\sin\phi}{p} - \frac{2h\sin\phi}{p^2}$,

$$\frac{\xi}{2}\frac{dU}{d\xi}+\frac{v}{2}\frac{dU}{v} = \frac{\cos^2\phi}{p^2} - \frac{ep\cos\phi+2h\cos^2\phi}{p^3} + \frac{\sin^2\phi}{p^2} - \frac{2h\sin^2\phi}{p^3}$$

$$= \frac{1}{p^2} - \frac{ep\cos\phi+2h}{p^3} = \frac{1}{p^2} - \frac{p+h}{} = -\frac{h}{p^3}, \text{ by (1).}$$

Hence the equations in $\xi$ and $v$ are equivalent to

$$\cos\phi - e - \frac{2h\cos\phi}{p} = -\frac{hx}{p^2}, \text{ and } \sin\phi - \frac{2h\sin\phi}{p} = -\frac{hy}{p^2}.$$

Substitute $p = \frac{h}{1-e\cos\phi}$, from (1); then these equations take the form

$$\frac{x}{\cos\phi-e\cos2\phi} = \frac{y}{\sin\phi-e\sin2\phi} = \frac{h}{(1-e\cos\phi)^2}.$$

Therefore the result of eliminating $\phi$ from these equations will be the same as that of eliminating $\xi$ and $v$ from the former equations. The labour of eliminating is much diminished by first transforming from $\phi$ to $\psi$ by means of the equation $\cos\phi = \frac{e+\cos\psi}{1+e\cos\psi}$; or the resulting equations in $\psi$ may be obtained directly and more speedily from the equations

$$x = p\cos\phi+u\sin\phi, \quad p = a(1+e\cos\psi),$$
$$y = p\sin\phi-u\cos\phi, \quad u = \frac{ebp\sin\psi}{h};$$

(See *Reprint*, Vol. XVI., p. 78.) By either method we obtain

$$x = \frac{h}{u^2}(\cos\psi-e\cos^2\psi+2e), \quad y = \frac{h}{u}\sin\psi(1-2e^2-e\cos\psi);$$

hence the result of eliminating $\psi$ from these equations, will be the same as that of eliminating $\xi$ and $v$ from the first group, or $\phi$ from the second group.

Put $\cos\psi = z$; then, observing that $a, b, e, h$ are the principal semiaxes, eccentricity, and semiparameter of the ellipse, we have, by successive substitutions, $aez^2 = az+2ae-x$, $\frac{b^2y^2}{a^2} = \frac{a}{e}(ae-aez^2)(1-2e^2-ez)^2$,

$$\frac{eb^2y^2}{a^2} = (x-ae-az)\left\{a(1-2e^2+4e^4)-ex+ae(4e^2-1)z\right\}$$
$$= (x-ae)\left\{a(1-2e^2+4e^4)-ex\right\} - a(4e^2-1)(2ae-x)$$
$$+ ae^2\left\{4ex-a(8e^2+1)\right\}z;$$

therefore $$z = \frac{(a^2x^2+b^2y^2)-(4e^2+3)\,a^3ex+(4e^4+6e^2-1)\,a^4}{\left\{4ex-(8e^2+1)\,a\right\}a^3e},$$

and $$az+2ae-x = \frac{a^2(1-4e^2)\,x^2+b^2y^2+2\,(6e^2-1)\,a^3ex-(12e^4-4e^2+1)\,a^4}{\left\{4ex-(8e^2+1)\,a\right\}a^2e}.$$

Hence the required eliminant is

$$\left\{(a^2x^2+b^2y^2)-(4e^2+3)\,a^3ex+(4e^4+6e^2-1)\,a^4\right\}^2 =$$
$$a^3\left\{4ex-(8e^2+1)a\right\}\left\{a^2(1-4e^2)x^2+b^2y^2+2\,(6e^2-1)a^3ex-(12e^4-4e^2+1)a^4\right\},$$

or $$(a^2x^2+b^2y^2)^2-2\,(5-4e^2)\,ea^5x^3-2\,(5+4e^2)\,ea^3b^2xy^2-(1-32e^2+16e^4)\,u^2a^6x^2$$
$$+(8e^4+20e^2-1)\,a^4b^2y^2+8\,(1-4e^2)\,eu^4a^7x-16e^2u^6a^8 = 0,$$

which, when transformed from $a$ and $b$ to $h$, gives the form in the question.

The elimination may be otherwise effected as follows:—

Let $x$, $y$ stand for $\frac{x}{h}$, $\frac{y}{h}$. Thus the two following linear equations in $z$ present themselves, between which $z$ must be eliminated,

$$ez^2-z+(u^2x-2e) = 0,$$
$$e^2z^4+(4e^3-2e)\,z^3+(4e^4-5e^2+1)\,z^2-(4e^3-2e)\,z+(u^6y^2-4e^4+4e^2-1) = 0.$$

Let these be written, *pro tem.*,

$$pz^2-qz+r = 0,\quad Az^4+Bz^3+Cz^2+Dz+E = 0.$$

Adopting the method of elimination by symmetrical functions, let $\alpha$, $\beta$ be the roots of the first equation; then the required eliminant will be the product of the two expressions

$$A\alpha^4+B\alpha^3+C\alpha^2+D\alpha+E = 0,\quad A\beta^4+B\beta^3+C\beta^2+D\beta+E = 0\ldots\ldots(2).$$

It consists of fifteen terms which are arranged on p. 95 with their values annexed. In computing those values the following quantities are required,

$$\left.\begin{aligned}
\alpha+\beta &= \frac{q}{p} = \frac{s_1}{p};\quad \alpha\beta = \frac{r}{p},\\
\alpha^2+\beta^2 &= \frac{q^2-2pr}{p^2} = \frac{s_2}{p^2},\\
\alpha^3+\beta^3 &= \frac{q\,(s_2-pr)}{p^3} = \frac{s_3}{p^3},\\
\alpha^4+\beta^4 &= \frac{qs_3-prs_2}{p^4} = \frac{s_4}{p^4},
\end{aligned}\right\}\quad
\begin{aligned}
s_1 &= 1,\\
s_2 &= -\,2eu^2x+(4e^2+1),\\
s_3 &= -\,3eu^2x+(6e^2+1),\\
s_4 &= 2e^2u^4x^2-(8e^3+4e)\,u^2x+(8e^4+8e^2+1).
\end{aligned}$$

$$\begin{aligned}
A &= e^2, & p &= e,\\
B &= 4e^3-2e, & q &= 1,\\
C &= 4e^4-5e^2+1, & r &= u^2x-2e,\\
D &= -\,4e^3+2e, & r^2 &= u^4x^2-4eu^2x+4e^2,\\
E &= u^6y^2-4e^4+4e^2-1, & r^3 &= u^6x^3-6eu^4x^2+12e^2u^2x-8e^3,\\
 & & r^4 &= u^8x^4-8eu^6x^3+24e^2u^4x^2-32e^3u^2x+16e^4.
\end{aligned}$$

In the following table the first column exhibits the product of equations (2), multiplied by $p^4$. The expansion of each term is placed opposite to it on the same line, the numbers being the numerical coefficients of the powers of $e$, $u$, $x$, $y$, which are written at the top of the respective columns.

| The fifteen terms of the Eliminant. | $u^8x^4$ | $u^{10}x^2y^2$ | $u^{12}y^4$ | $u^6x^3$ | $u^8xy^2$ | $u^4x^2$ | $u^6y^2$ | $u^2x$ | 1 |
|---|---|---|---|---|---|---|---|---|---|
| | $e^4$ | $e^4$ | $e^4$ | $e^7$ $e^5$ $e^3$ | $e^7$ $e^5$ $e^3$ | $e^{10}$ $e^8$ $e^6$ $e^4$ $e^2$ | $e^8$ $e^6$ $e^4$ $e^2$ | $e^{11}$ $e^9$ $e^7$ $e^5$ $e^3$ | $e^{12}$ $e^{10}$ $e^8$ $e^6$ $e^4$ $e^2$ |
| $A^2r^4$ = | +1 | ... | ... | − 8 | ... | +24 | ... | − 32 | + 16 |
| $+ABr^3s_1$ = | ... | ... | ... | + 4−2 | ... | −24+12 | ... | + 48−24 | − 32+16 |
| $+ACr^2s_2$ = | ... | ... | ... | − 8+10−2 | ... | +48−56+ 7+1 | ... | − 96+104− 4−4 | + 64− 64− 4+4 |
| $+ADrs_3$ = | ... | ... | ... | ... | ... | +12− 6 | ... | − 48+20+2 | + 48−16−4 |
| $+AEs_4$ = | ... | +2 | ... | ... | − 8−4 | − 8+ 8− 2 | + 8+8+1 | + 32− 16− 8+4 | − 32 +20−4−1 |
| $+B^2pr^3$ = | ... | ... | ... | +16−16+4 | ... | −96+96−24 | ... | +192−192+48 | −128+128−32 |
| $+BCpr^2s_1$ = | ... | ... | ... | ... | ... | +16−28+14−2 | ... | − 64+112−56+8 | + 64−112+56−8 |
| $+BDprs_2$ = | ... | ... | ... | ... | ... | +32−32+ 8 | ... | −128+112−16−4 | +128− 96 +8 |
| $+BEps_3$ = | ... | ... | ... | ... | −12+6 | ... | +24−8−2 | + 48− 72+36−6 | − 96+128−48 +2 |
| $+C^2p^2r^2$ = | ... | ... | ... | ... | ... | +16−40+33−10+1 | ... | −64+160−132+40−4 | +64−160+132−40+4 |
| $+CDp^2rs_1$ = | ... | ... | ... | ... | ... | ... | ... | − 16+ 28−14+2 | + 32− 56+28−4 |
| $+CEp^2s_2$ = | ... | ... | ... | ... | −8+10−2 | ... | +16−16−1+1 | +32− 72+ 56−18+2 | −64+128− 76+ 8+5−1 |
| $+D^2p^3r$ = | ... | ... | ... | ... | ... | ... | ... | + 16− 16+ 4 | − 32+ 32−8 |
| $+DEp^3s_1$ = | ... | ... | ... | ... | ... | ... | − 4+2 | ... | + 16− 24+12−2 |
| $+E^2p^4$ = | ... | ... | +1 | ... | ... | ... | − 8+ 8−2 | ... | +16− 32+ 24− 8+1 |
| | +1 | +2 | +1 | + 8−10 | −8−10 | +16−48+33− 1 | + 8+20−1 | −32+ 72− 48+ 8 | +16− 48+ 48−16 |

Dividing the result by $e^4u^6$, and substituting for $x$ and $y$ their values, viz., $\frac{x}{h}$ and $\frac{y}{h}$, we obtain in this way, for the eliminant, the value given in the question, which agrees with the result given by the EDITOR, on p. 78 of Vol. XVI. of the *Reprint*.

A useful test of the accuracy of the result is afforded by substituting the simultaneous values $e=\frac{1}{2}$, $u^2=\frac{3}{4}$, $x=\frac{1}{2}$, $y=1$, $h=1$.

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**3757.** (Proposed by Miss LADD.)—In a quadrilateral ABCD, join the centres of gravity of the triangles ABC, ACD, ABD, BCD, and show, by Quaternions, that the resulting quadrilateral will be similar to ABCD, and that each side will be equal to one third of the corresponding side of ABCD.

*Solution by the* PROPOSER; *and others.*

Call the centres of gravity $a, b, c, d$; $a$ belonging to BCD. Take A for origin, and let the vectors of B, C, D be $\beta, \gamma, \delta$; then the vectors of $a, b, c, d$ are $\frac{1}{3}(\beta+\gamma+\delta)$, $\frac{1}{3}(\gamma+\delta)$, $\frac{1}{3}(\beta+\delta)$, $\frac{1}{3}(\beta+\gamma)$, or of $bcd$ referred to $a$, $-\frac{1}{3}\beta$, $-\frac{1}{3}\gamma$, $-\frac{1}{3}\delta$. The versors of these vectors are the same as of BCD referred to A, but reversed; therefore $abcd$ is similar to ABCD, and has its sides parallel; also the tensors of $bcd$ are one-third of the tensors of BCD; therefore the sides of $abcd$ are one-third of $abcd$.

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**3680.** (Proposed by Professor TOWNSEND.)—If $a$, $b$, $c$ be the three semi-axes of a solid ellipsoid of uniform density $\rho$, and A, B, C the three functions of them, which, multiplied respectively by the three corresponding coordinates $x$, $y$, $z$ of any particle $\mu$ internal to its mass, give the three components X, Y, Z of its attraction on the particle for the law of inverse square of distance; show that $\frac{\text{A}}{a}\,da + \frac{\text{B}}{b}\,db + \frac{\text{C}}{c}\,dc$ is an exact differential.

*Solution by the* PROPOSER.

For since, adopting the ordinary notation,

$$\text{A} = -\frac{\mu f\rho}{a^2}\cdot\iint\frac{\cos^2\alpha\,d\omega}{\left[\frac{1}{a^2}\cos^2\alpha+\frac{1}{b^2}\cos^2\beta+\frac{1}{c^2}\cos^2\gamma\right]},$$

$$\text{B} = -\frac{\mu f\rho}{b^2}\cdot\iint\frac{\cos^2\beta\,d\omega}{\left[\frac{1}{a^2}\cos^2\alpha+\frac{1}{b^2}\cos^2\beta+\frac{1}{c^2}\cos^2\gamma\right]},$$

$$\text{C} = -\frac{\mu f\rho}{c^2}\cdot\iint\frac{\cos^2\gamma\,d\omega}{\left[\frac{1}{a^2}\cos^2\alpha+\frac{1}{b^2}\cos^2\beta+\frac{1}{c^2}\cos^2\gamma\right]};$$

therefore at once

$$\frac{d}{dc}\left[\frac{B}{b}\right] = \frac{d}{db}\left[\frac{C}{c}\right] = \frac{2\mu f\rho}{b^3c^3}\cdot\iint\frac{\cos^2\beta\cos^2\gamma\,d\omega}{\left[\frac{1}{a^2}\cos^2\alpha+\frac{1}{b^2}\cos^2\beta+\frac{1}{c^2}\cos^2\gamma\right]^2},$$

$$\frac{d}{da}\left[\frac{C}{c}\right] = \frac{d}{dc}\left[\frac{A}{a}\right] = \frac{2\mu f\rho}{c^3a^3}\cdot\iint\frac{\cos^2\gamma\cos^2\alpha\,d\omega}{\left[\frac{1}{a^2}\cos^2\alpha+\frac{1}{b^2}\cos^2\beta+\frac{1}{c^2}\cos^2\gamma\right]^2},$$

$$\frac{d}{db}\left[\frac{A}{a}\right] = \frac{d}{da}\left[\frac{B}{b}\right] = \frac{2\mu f\rho}{a^3b^3}\cdot\iint\frac{\cos^2\alpha\cos^2\beta\,d\omega}{\left[\frac{1}{a^2}\cos^2\alpha+\frac{1}{b^2}\cos^2\beta+\frac{1}{c^2}\cos^2\gamma\right]^2};$$

and therefore, &c.

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**3686.** (Proposed by M. Collins, B.A.)—To find convenient formulæ for approximating to the square root of any rational number N.

*Solution by the* Proposer.

Let $N^{\frac{1}{2}} = x$ nearly, and $= x+d$ exactly, where $d$ is small and $< \frac{1}{2}$, then we have $\quad d^3 = (N^{\frac{1}{2}}-x)^3 = N^{\frac{3}{2}} - 3Nx + 3N^{\frac{1}{2}}x^2 - x^3$;

therefore $\quad N^{\frac{1}{2}} = \dfrac{(3N+x^2)\,x+d^3}{3x^2+N} = \dfrac{(3N+x^2)\,x}{3x^2+N} + \dfrac{d^3}{4N}$ very nearly

(since $x^2 = N$ nearly), a result which, with the second fraction omitted, gives Hutton's approximation.

Again, $\quad d^5 = (N^{\frac{1}{2}}-x)^5 = N^{\frac{5}{2}} - 5N^2x + 10N^{\frac{3}{2}}x^2 - 10Nx^3 + 5N^{\frac{1}{2}}x^4 - x^5$;

therefore $\quad N^{\frac{1}{2}} = \dfrac{x\,(x^4+10Nx^2+5N^2)+d^5}{N^2+10Nx^2+5x^4} = \dfrac{5\,(N+x^2)^2-(2x^2)^2}{5\,(N+x^2)^2-(2N)^2}\cdot x + \dfrac{d^5}{(4N)}$

very nearly (since $x^2 = N$ nearly), a result which is nearly true with the second fraction omitted, and affords an easy and elegant method for finding a rational fraction very nearly equal to the square root of any rational number N.

For example, to find $3^{\frac{1}{2}}$, take $x = 2$; then since here $N = 3$, we find

$$3^{\frac{1}{2}} = \frac{5\times7^2-8^2}{5\times7^2-6^2}\times 2 = \frac{362}{209} \text{ nearly; in fact } \left(\frac{362}{209}\right)^2 = 3\,\frac{1}{43681}.$$

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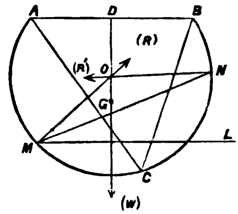
**3634.** (Proposed by J. J. Walker, M.A.)—An arc of rigid uniform circular hoop is to be placed in a position of equilibrium, with its convex side resting on two horizontal pins fixed in a vertical wall. If there is no friction, show that, for such a position to be possible, the inclination (to the horizontal) of the line joining the pins must not be greater than the less of the two angles ABC, BAC; AB being the chord of the arc, and AC the chord, equal in length to that line, of a portion of the arc. Determine the pressures when this condition is satisfied.

*Solution by* W. SIVERLEY; A. B. EVANS, M.A.; *and others.*




Let R, R′ be the reaction of the pins M, N; G the centre of gravity; and W the weight of the arc AMCNB. Put NML $= \phi =$ the inclination of NM to the horizontal, OMN $=$ ONM $= \theta$.

Since the arc is kept in equilibrium by the forces R, R′, and W, the directions of these three forces must pass through the centre O, and AB must be parallel to ML. Since AC $=$ MN, the angle ABC is equal to the maximum value of NML consistent with equilibrium, and as B cannot fall below N, the angle BAC reaches its minimum value, consistent with equilibrium, when it becomes equal to NML; therefore NML is never greater than the less of the two angles ABC, BAC.

To obtain the pressures, resolve horizontally, and take moments about M; then $R\cos(\theta+\phi) = R'\cos(\theta-\phi)$, and $R'\sin 2\theta = W\cos(\theta+\phi)$;

whence $$R' = W\frac{\cos(\theta+\phi)}{\sin 2\theta}, \text{ and } R = W\frac{\cos(\theta-\phi)}{\sin 2\theta}.$$

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**3672.** (Proposed by Professor BALL.)—A plane figure is moving in any manner in a plane. Every point of the figure describes a curve. Show that all the points upon the circumference of a certain circle are at any instant situated in points of inflexion of the curves which they describe. Show, also, that two points on the circle are describing straight lines more nearly than any other points.

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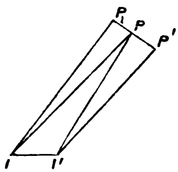
*Solution by* R. W. GENESE, B.A.




Let I be the instantaneous centre of rotation at any time $t$. If $\omega$ be the angular velocity about I, in time $dt$ any point $P_1$ will move to P so that $P_1P = IP\omega dt$; in the same time let I′ be the new centre of rotation, then $\frac{II'}{dt} =$ vel. of centre of rotation $=$ $v$, say.

If P is to continue in the same straight line; *i.e.* if it be a point of inflexion of its locus, PI′ must be parallel to $P_1I$; therefore angle $P_1IP =$ angle IPI′, $\omega dt = v dt \sin\phi \div r$, where $r =$ IP, $\phi =$ angle $r$ makes with II′, or $r = \frac{v}{\omega}\sin\phi$.

The locus of P is thus a circle of diameter $\frac{v}{\omega}$ touching II′. The two points of this circle which most nearly describe a straight line are clearly its intersections with its consecutive position.

One of these is the point I itself, the other is on the straight line drawn from I perpendicular to the locus of the centre of the circle.

This locus makes an angle with II′ whose tangent is

$$d\left(\frac{v}{2\omega}\right) + \left(\mathrm{R} - \frac{v}{2\omega}\right)\frac{vdt}{\mathrm{R}},$$

where R is the radius of curvature of locus of I.

[This is the geometrical theory of "Parallel Motions" in mechanism.]

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**3479.** (Proposed by J. B. Sanders.)—A body descending vertically draws an equal body 25 feet in $2\frac{1}{2}$ seconds, up a plane inclined 30° to the horizon, by means of a string passing over a pully at the top of the plane. Determine the force of gravity.

*Solution by* Millicent Colquhoun.

Let each of the masses be unity; then the mass moved is 2, and the accelerating force is $f = \frac{1}{2}g\,(1-\frac{1}{2}) = \frac{1}{4}g$; hence, from the formula $s = \frac{1}{2}ft^2$, we have $$g = 25 \times 8 \div (2\tfrac{1}{2})^2 = 32.$$

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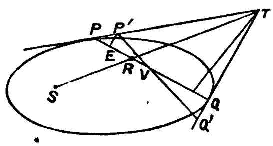
**3529.** (Proposed by the Rev. A. F. Torry, M.A.)—Two particles, P, Q, describe the same orbit round the same centre of force S; their directions of motion intersect in T, ST cuts PQ in R, and the ultimate intersection of consecutive positions of PQ is U: prove that PU = QR, and that when PQ is a maximum ST cuts it at right angles.

I. *Solution by* Professor Wolstenholme.




Supposing the points P, Q instantaneously to move along the tangents at P, Q to P′, Q′, and draw P′E parallel to TQ; then $\dfrac{\mathrm{P'E}}{\mathrm{PP'}} = \dfrac{\mathrm{TQ}}{\mathrm{TP}}$, $\dfrac{\mathrm{P'E}}{\mathrm{QQ'}} = \dfrac{\mathrm{P'V}}{\mathrm{VQ}}$; if V be the point of intersection of PQ, P′Q′, whence $\dfrac{\mathrm{PP'}}{\mathrm{QQ'}} = \dfrac{\mathrm{TP}}{\mathrm{TQ}}\cdot\dfrac{\mathrm{P'V}}{\mathrm{VQ'}}$.

But $\triangle\mathrm{SPP'} = \triangle\mathrm{SQQ'}$; wherefore $\dfrac{\mathrm{PP'}}{\mathrm{QQ'}} = \dfrac{\sin\mathrm{STQ}}{\sin\mathrm{STP}} = \dfrac{\mathrm{QR}}{\mathrm{TQ}}\cdot\dfrac{\mathrm{TP}}{\mathrm{RP}}$;

whence $\dfrac{\mathrm{P'V}}{\mathrm{VQ'}} = \dfrac{\mathrm{TP}}{\mathrm{RP}}$, or taking the limit $\dfrac{\mathrm{PU}}{\mathrm{UQ}} = \dfrac{\mathrm{QR}}{\mathrm{RP}}$; $\therefore$ PU = RQ.

When PQ is a maximum or minimum PQ = P′Q′,

$$\mathrm{PT}^2 + \mathrm{QT}^2 - 2\mathrm{PT}\,.\,\mathrm{QT}\cos\mathrm{PTQ} = \mathrm{P'T}^2 + \mathrm{Q'T}^2 - 2\mathrm{P'T}\,.\,\mathrm{Q'T}\cos\mathrm{PTQ},$$

whence $$\mathrm{PP'}\,(\mathrm{PT} - \mathrm{QT}\cos\mathrm{PTQ}) = \mathrm{QQ'}\,(\mathrm{QT} - \mathrm{PT}\cos\mathrm{PTQ}),$$

or $$\mathrm{PP'} \times \mathrm{PQ}\cos\mathrm{QPT} = \mathrm{QQ'}\,.\,\mathrm{PQ}\cos\mathrm{PQT};$$

or $$\mathrm{PP'} \times \frac{\mathrm{PW}}{\mathrm{PT}} = \mathrm{QQ'}\,.\,\frac{\mathrm{QW}}{\mathrm{QT}},$$ if TW be perpendicular to PQ;

$$\text{or} \qquad \frac{PP'}{QQ'} = \frac{PT}{QT} \cdot \frac{QW}{PW}; \text{ therefore } \frac{QW}{PW} = \frac{QR}{PR};$$

or W coincides with R, *i.e.*, ST is perpendicular to PQ.

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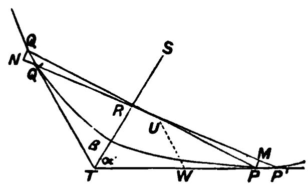
II. *Solution by* G. S. CARR.




1. Let P′, Q′ be the positions of the particles (which we will suppose to be moving in the same direction round S) after a small interval of time has elapsed; $\angle RTP = \alpha$, $RTQ = \beta$.

Then ultimately, by the theory of central forces, PP′ and QQ′ are as the velocities, and inversely as the perpendiculars from S upon the tangents, at P and Q; therefore $\quad PP' : QQ' = \sin\beta : \sin\alpha$.

Draw PM, QN perpendicular to PQ; then

$$\frac{PU}{QU} = \frac{PM}{QN} = \frac{PP' \sin P}{QQ' \sin Q} = \frac{\sin\beta \sin P}{\sin\alpha \sin Q} = \frac{QR}{PR};$$

therefore $\qquad PU = QR.$

*Otherwise*:—By the theory of envelopes; let

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

be the equation to PQ referred to the tangents as axes.

Differentiating for $a$ we have $\quad -\frac{x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(2)},$

$\frac{db}{da}$ = ratio of velocities at P and Q = inverse ratio of perpendiculars on tangents $= -\frac{\sin\alpha}{\sin\beta}$ (minus because $b$ diminishes as $a$ increases). Substitute this value, and solve (1) and (2) for $x$ and $y$, the coordinates of the point of ultimate intersection. We obtain

$$\frac{x}{a} = \frac{a\sin\alpha}{a\sin\alpha + b\sin\beta}; \text{ therefore } \frac{TW}{TP} = \frac{QU}{QP} = \frac{PR}{PR+RQ} = \frac{PR}{QP};$$

therefore $\qquad QU = PR.$

2. Again when PQ is a maximum, the increment P′M = the decrement Q′N, that is, $\quad PP' \cos P = QQ' \cos Q$; therefore $\frac{QQ'}{PP'} = \frac{\cos P}{\cos Q} = \frac{\sin\alpha}{\sin\beta}$; therefore ST is at right angles to PQ, for only one line can divide the angle PTQ so that $\sin\alpha : \sin\beta = \cos P : \cos Q$.

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**3717.** (Proposed by T. COTTERILL, M.A.)—If the lines of the pentagon *adbec* touch a parabola, show that the area of the pentagon *abcde* vanishes.

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*Solution by the* PROPOSER.

It follows from the solution of Question 3674, that if $p$ be the perpen-

dicular from a point $p$ on an arbitrary line of the plane, and $\pi$ the product of the perpendiculars of the fixed points $a, b, c, d, e$ on the same line, then

$$\frac{\pi}{p}\left\{\frac{(pab)}{a.b}+\frac{(pbc)}{b.c}+\frac{(pcd)}{c.d}+\frac{(pde)}{d.e}+\frac{(pea)}{e.a}\right\}=\Sigma,$$

is independent of the point $p$. Consequently by giving $p$ the positions $a \ldots e$, we have five different integral forms. Thus

$$\Sigma=(abc)\,d.e+(acd)\,e.b+(ade)\,b.c$$
$$=(bcd)\,e.a+(bde)\,a.c+(bea)\,c.d=\&c.$$

Hence $\Sigma=0$ is the class equation of the conic touching the lines of the pentagon *adbec*. If the conic is a parabola, the line at infinity is a tangent, in which case $a=b=c=d=e$, so that $(abc)+(acd)+(ade)=0$, or the area *abcde* vanishes.

Cor. $\Sigma\dfrac{(pab)}{a.b}=0$ identically, if the arbitrary line is a tangent to the conic, and in any other case is the pentagonal equation to it.

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**3722.** (Proposed by J. W. L. Glaisher, B.A., F.R.A.S.)—Prove that

$$\Sigma\Sigma\frac{1}{(x+i)(x+j)}=-\pi^2,$$

$i$ and $j$ having all integer values from $-\infty$ to $\infty$ (including zero), subject to the condition that in no term must $i$ and $j$ have the same value. ($x$ must not itself be an integer.)

*Solution by the* Proposer.

$$\operatorname{Cot}x=\frac{1}{x}+\frac{1}{x-\pi}+\frac{1}{x+\pi}+\frac{1}{x-2\pi}+\frac{1}{x+2\pi}+\ldots,$$

whence $$\pi\cot\pi x=\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x+1}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

By differentiation $$\pi^2\operatorname{cosec}^2\pi x=\frac{1}{x^2}+\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}+\ldots,$$

and squaring (1),

$$\pi^2\cot^2\pi x=\frac{1}{x^2}+\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}+\ldots+\Sigma\Sigma\frac{1}{(x+i)(x+j)};$$

whence by subtraction $$\pi^2+\Sigma\Sigma\frac{1}{(x+i)(x+j)}=0.$$

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**3730.** (Proposed by S. Roberts, M.A.)—If I is an invariant of $(a_0, a_1, a_2, \ldots a_p)(x, y)^p$, and is of the degree $k$ in the coefficients, then, writing

$\Delta$ for $$na_1\frac{d}{da_0}+(n-1)\,a_2\frac{d}{da_1}+\ldots+a_n\frac{d}{da_{n-1}},$$

we have $$\frac{\Delta^{k(n-p)}}{1.2\ldots k(n-p)}\mathrm{I} = \mathrm{I}', \qquad n>p,$$
where I′ is an invariant of $(a_{n-p}, a_{n-p+1}, \ldots a_n)(x, y)^p$, and is of the order $k$ in the coefficients.

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*Solution by the* PROPOSER.

For the invariant I is the leading term of a covariant of the quantic
$$(a_0, a_1, \ldots a_n)(x, y)^n, \qquad n>p,$$
and the covariant is expressed by
$$\mathrm{I}x^{k(n-p)} + \Delta \mathrm{I}x^{k(n-p)-1}y + \frac{\Delta^2\mathrm{I}}{1.2}x^{k(n-p)-2}y^2 + \ldots + \frac{\Delta^{k(n-p)}}{1.2\ldots k(n-p)}\mathrm{I}y^{k(n-p)}.$$
Hence $\dfrac{\Delta^{k(n-p)}}{1.2\ldots k(n-p)}\mathrm{I}$ is the same as I with $a_n$, $a_{n-1}$, &c. written instead of $a_0$, $a_1$, &c. Therefore, since I is an invariant of $(a_0, \ldots a_p)(x, y)^p$, I′ is a like invariant of $(a_n, a_{n-1}, \ldots a_{n-p})(x, y)^p$, which is tantamount to the statement in the question. [See Question 3401.]

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**3220.** (Proposed by J. J. WALKER, M.A.)—The theorem in Question 3122 may be generalized by supposing that PO bears any constant ratio ($k$) to the conjugate semi-diameter.

1. Prove that the circle passing through LMN, the point P referred to the axes of the ellipse being $(x', y')$, is
$$x^2+y^2 - \frac{b(b-ka)}{a^2}x'x - \frac{a(a-kb)}{b^2}y'y - (a^2+b^2-kab) = 0.$$

2. Verify that this circle passes through the point on the ellipse diametrically opposite to P; and find its envelope as P moves round the ellipse.

3. If the normals at L, M, N meet the ellipse which is the locus of O again in L′, M′, N′, prove that LL′, MM′, NN′ are to the semi-diameters parallel to the tangents at L, M, N respectively as $k$ to 1.

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*Solution by the* PROPOSER.

1. Writing $m = a-kb$, $n = b-ka$, the coordinates of L, M, N must satisfy the three equations $b^2x^2+a^2y^2-a^2b^2 = 0$ .......................... (1),
$bny'xy - amx'y^2 + c^2xy^2 = 0$ or $b^2ny'(amx'y - bny'x) - ac^2mx'y^2 + c^4xy^2 = 0$ ...(2),
and $\qquad c^4x^3 - (am+bn)c^2x'x^2 + a^3m(2bn-am)x + a^4mx' = 0$ ...........(3);
(2), because the normal at $xy$ passes through $\mathrm{O}\left(\dfrac{mx'}{a}, \dfrac{ny'}{b}\right)$. It will be found that $(1)\times\left\{c^4x + (b^3n - ac^2m)x'\right\} - (2)\times a^2 - (3)\times b^2$, after division by $a^2b^3n$, reduces to the equation in (1).

2. For the envelope of this circle we have

$$\frac{bmxdx'}{a^2} + \frac{anydy'}{b^2} = 0, \quad \frac{x'dx'}{a^2} + \frac{y'dy'}{b^2} = 0,$$

whence $amyx' - bnxy' = 0$. From this and the equation to the circle, $x'y$, are found linearly; and substituting in $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1 = 0$, the irrelevant factor $\frac{b^2n^2x^2}{a^2} + \frac{a^2m^2y^2}{b^2}$ being rejected, the required envelope is found to be

$$(x^2+y^2)^2 - \frac{a^4+(a^2+bn)^2}{a^2}x^2 - \frac{b^4+(b^2+am)^2}{b^2}y^2 + (a^2+b^2-kab)^2 = 0.$$

3. Let the running coordinates of the normal at $(x, y)$ be $\xi, \eta$; then, for the intersections of the normal with the locus of O, viz., $\frac{\xi^2}{m^2} + \frac{\eta^2}{n^2} - 1 = 0$, we find a quadratic, one root of which is, of course, $\xi = \frac{mx}{a}$; the other is $\xi = \frac{c^4x^2 + am^3(2bn - am)}{c^2(am+bn)x^2 - a^2m^2}x$. But, from (3), $\frac{mx'}{a}$ (*i.e.* the abscissa of O) = this latter value of $\xi$; consequently L′ is the intersection for which $\xi = \frac{mx}{a}$, *i.e.* L′ is the point such that LL′ is to the conjugate semi-diameter as $k$ to 1.

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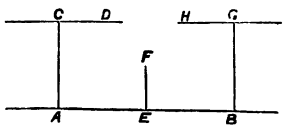
## Proof of a Fundamental Property of Parallel Straight Lines.

*By* J. Walmsley, B.A.

Prop. I.—*Any straight line perpendicular to one of two parallel straight lines will meet the other.*




Let AB, CD be the parallels; and EF be perpendicular to AB: EF will meet CD.

For, if not, it is parallel to CD.

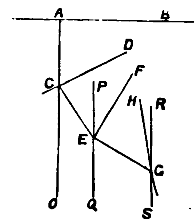
Draw CA perpendicular to AB, and therefore parallel to BE (Euc. I. 28). Make EB equal to B, the angle EBG equal to EAC, BG equal to AC, and the angle BGH equal to ACD. Then FEACD may be superposed upon FEBGH. Therefore GH (like CD) is parallel to AB and also to EF.

Now EF cuts AB at E, and cannot therefore cut it again; and it cannot cut AC, CD, BG, or GH, being parallel to each of them. Therefore it lies between CD and GH. But AB cuts EF and is parallel to both CD and GH. Therefore AB, also, lies between CD and GH; which is absurd, since CD and GH were constructed on the same side of AB.

Prop. II.—*Any straight line which is perpendicular to one of two parallel straight lines is perpendicular to the other also.*

For, *if not*, let ACO be perpendicular to AB, one of two parallel straight

lines, but not to CD, the other; and let ACD be the acute angle of those which are adjacent at C. Make the angle DCE equal to the angle BAC, CE equal to AC, and the angle CEF equal to ACD; so that the system of lines DCEF may be superposed upon the system BACD. Similarly construct FEGH so as to be superposable upon BACD. Then EF is parallel to CD, and therefore also to AB; for it cannot meet AB without first cutting CD. Similarly GH is parallel to AB; and so is every other line of the series CD, EF, &c., obtained by continuing the construction.




Make the angle CEP equal to ECO, and produce PE to Q; also EGR equal to GEQ and produce RG to S. Then PQ is parallel to AO (Euc. I. 27); and RS is also parallel to AO; for it is parallel to PQ, and cannot meet AO without first cutting PQ. Similarly, if the construction be continued, every other line of the series PQ, RS, &c. will be parallel to AO, so long as they fall successively farther away from AO.

Now FEQ and FEC are together greater than two right angles by the angle PEC, and therefore by the angle OCE; while FEG and FEC, which are equal to DCE and DCA, are together *less* than two right angles by the angle OCE Therefore the angle QEG is equal to twice the angle OCE. In the same way it may be shown that the next angle of the series OCE, QEG, &c., would be thrice the angle OCE, and so on.

Now the angle OCE, if less than the angle ACD, can always be multiplied so as to exceed it; and the foregoing construction may be continued (Euc. I. 23, and post. 2) until any (finite) number of angles of the series OCE, QEG, &c., have been obtained; and therefore until one of them occurs greater than the angle ACD. Whichever angle of the series this may be will not affect the remainder of the argument; so let us suppose it the angle QEG. Then the angle QEG is also greater than the angle EGH. Wherefore GH falls within the angle EGR. Now GH cannot cut either GS or GE at any other point than G; and it cannot meet EF, being parallel to it; therefore it cannot meet OA. Hence OA, a perpendicular to AB, does not meet GH, which is parallel to AB; which is impossible, by Prop. I.

[Since this has been in type Mr. Walmsley writes that he has observed a simpler mode of exhibiting the argument of Prop. I., which will perhaps be obvious to most readers; and also that it seems easy to replace its axiomatic basis by a simpler one. The following is one of several which could be used for the purpose:—*If three straight lines be parallel to each other there must be some one straight line at least which cuts all three.*]

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**3711.** (Proposed by Professor Townsend.)—If $\tau_1, \tau_2, \tau_3$ be the three differences between the sidereal times of the true and apparent transits of three stars whose declinations are $\delta_1, \delta_2, \delta_3$, as observed through a transit telescope at any latitude $\lambda$; show that, whatever be the errors $x$

and $y$ of the instrument in deviation and level, its collimation error $z$ is given by the formula

$$\frac{\tau_1 \cos \delta_1 \sin (\delta_2-\delta_3)+\tau_2 \cos \delta_2 \sin (\delta_3-\delta_1)+\tau_3 \cos \delta_3 \sin (\delta_1-\delta_2)}{\sin (\delta_2-\delta_3)+\sin (\delta_3-\delta_1)+\sin (\delta_1-\delta_2)};$$

which, as not containing $\lambda$ explicitly, is consequently independent of the latitude of the place of observation.

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*Solution by the* PROPOSER.

For since, for the determination of the three aforesaid errors $x, y, z$ of the instrument, we have (see GODFRAY'S *Astronomy*, Art. 82) the three linear equations

$$x \cos (\delta_1+\lambda)+y \sin (\delta_1+\lambda)+z = \tau_1 \cos \delta_1,$$
$$x \cos (\delta_2+\lambda)+y \sin (\delta_2+\lambda)+z = \tau_2 \cos \delta_2,$$
$$x \cos (\delta_3+\lambda)+y \sin (\delta_3+\lambda)+z = \tau_3 \cos \delta_3;$$

therefore, multiplying respectively by $\sin (\delta_2-\delta_3)$, $\sin (\delta_3-\delta_1)$, $\sin (\delta_1-\delta_2)$, and adding, $x$ and $y$ become eliminated, and there remains for $z$ the value above given.

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3731. (Proposed by S. WATSON.)—EG is a focal chord in an ellipse, and MP a perpendicular to it at its middle point M. Find the form and area of the curve which the line MP always touches.

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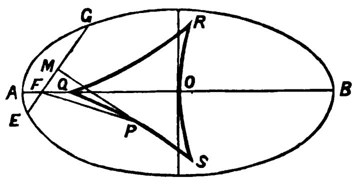
I. *Solution by the* Rev. Dr. BOOTH, F.R.S.




Let $r$ be the distance of the middle point of the chord from the focus, then it is easily shown that

$$r = \frac{a(1-e^2)\, e \cos \theta}{1-e^2 \cos^2 \theta};$$

but $\quad r = (ae-\xi^{-1}) \cos \theta,$

and $\quad \cos \theta = \dfrac{\xi}{(\xi^2+v^2)^{\frac{1}{2}}}.$

Hence eliminating, the tangential equation of the curve becomes

$$a^2v^2+b^2\xi^2 = a^3e^3\xi v^2.$$

Separating the variables and solving for $v$, we have $v = \dfrac{b\xi}{a(ae\xi-1)^{\frac{1}{2}}}$.

Hence, giving a series of values to $\xi$, we may find the corresponding values for $v$, and thus rule out the curve, which is of the form shown in the diagram. Its rectification presents no difficulty, for $p = \dfrac{a^3e^3 \cos \lambda \sin^2 \lambda}{a \sin^2 \lambda + b^2 \cos^2 \lambda}$.

The projective equation of the foot of the perpendicular from the centre on a tangent to this curve is $(a^2y^2+b^2x^2)(x^2+y^2) = a^3e^3xy^2$.

II. *Solution by the* PROPOSER.

Let F be the focus, $FP = \rho$, $\angle GFO = \theta$, $\angle PFO = \phi$; then

$$\rho\cos(\theta+\phi) = FM = \frac{b^2}{2}\left(\frac{1}{a-c\cos\theta} - \frac{1}{a-c\cos\theta}\right) = \frac{b^2c\cos\theta}{a^2-c^2\cos^2\theta} \;\ldots\ldots(1),$$

and differentiating with respect to $\theta$, we have

$$\rho\sin(\theta+\phi) = \frac{b^2c\,(a^2+c^2\cos^2\theta)\sin\theta}{(a^2-c^2\cos^2\theta)^2} \;\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$(1)^2+(2)^2$ gives $$\rho^2 = \frac{b^4c^2\left\{(a^2-c^2\cos^2\theta)^2\cos^2\theta + (a^2+c^2\cos^2\theta)^2\sin^2\theta\right\}}{(a^2-c^2\cos^2\theta)^4}\;\ldots(3).$$

$(2)\div(1)$ gives $$\theta+\phi = \tan^{-1}\left\{\frac{(a^2+c^2\cos^2\theta)\sin\theta}{(a^2-c^2\cos^2\theta)\cos\theta}\right\};$$

therefore $$d\theta+d\phi = \frac{(a^4-4a^2c^2\sin^2\theta\cos^2\theta-c^4\cos^4\theta)\,d\theta}{(a^2-c^2\cos^2\theta)^2\cos^2\theta+(a^2+c^2\cos^2\theta)^2\sin^2\theta},$$

and $$d\phi = -\frac{2c^2\,(3a^2\sin^2\theta-b^2\cos^2\theta)\cos^2\theta\,d\theta}{(a^2-c^2\cos^2\theta)^2\cos^2\theta+(a^2+c^2\cos^2\theta)^2\sin^2\theta}.$$

It is easily seen the curve is of the form given in the diagram, where $FQ = \dfrac{b^2c}{a^2}$, and the principal axes of the ellipse are tangents to the curve at the cusp Q, and the centre O. Half the area of the curve lies between the limits $\theta=0$, $\theta=\frac{1}{2}\pi$; hence the whole area is

$$\int\rho^2d\phi = -2b^4c^4\int_0^{\frac{1}{2}\pi}\frac{\left\{3a^2\sin^2\theta-b^2\cos^2\theta\right\}\cos^2\theta\,d\theta}{(a^2-c^2\cos^2\theta)^4},$$

$$= -2b^4c^4\int_0^{\infty}\frac{\left\{3a^2z^4+(3a^2-b^2)z^2-b^2\right\}dz}{(b^2+a^2z^2)^4}$$

(where $z=\tan\theta$) $= \dfrac{\pi e^3c^3}{8b} = \dfrac{e^4c^2}{8b^2}$ of the given ellipse.

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**3712.** (Proposed by Professor WOLSTENHOLME.)—On the normal at a point P of an ellipse is drawn $PQ=CD$ (inwards), and from Q are drawn the three other normals QL, QM, QN; prove that (1) the tangents at L, M, N form a triangle whose circumscribed circle is concentric with the ellipse; (2) the nine points' circle of this triangle touches both the auxiliary circles of the ellipse; and (3) meets the ellipse in three points L′, M′, N′, forming a triangle whose inscribed circle is concentric with the ellipse; (4) the distance between the centres of gravity of the triangles LMN, L′M′N′ is bisected by the centre of the ellipse; (5) if $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be the excentric angles of L, M, N, L′, M′, N′; then $\alpha'+\beta'+\gamma' = \alpha+\beta+\gamma$ (or of course $2\pi+\alpha+\beta+\gamma$), and if either of these be denoted by $\delta$, then $(\alpha, \beta, \gamma)$, $(\alpha', \beta', \gamma')$ are the three roots, respectively, of the equations $\tan\theta = \dfrac{b}{a}\tan\frac{1}{2}(\theta-\delta)$, $\tan\theta = \dfrac{-a}{b}\tan\frac{1}{2}(\theta-\delta)$.

*Solution by the* PROPOSER.

If $\theta$ be the excentric angle of the point P, the coordinates of Q are $(a-b)\cos\theta$, $(b-a)\sin\theta$; and if $\phi$ be the excentric angle of L, M, or N,

then $$a(a-b)\frac{\cos\theta}{\cos\phi}+b(a-b)\frac{\sin\theta}{\sin\phi}=a^2-b^2;$$

or, rejecting the factor $\sin\frac{1}{2}(\theta-\phi)$, $\tan\phi\tan\frac{1}{2}(\phi+\theta)=\frac{b}{a}$,

whence, if $\alpha$, $\beta$, $\gamma$ be the three roots of this equation,

$$\tan\frac{\beta-\gamma}{2}=\tan\left(\frac{\beta+\theta}{2}-\frac{\gamma+\theta}{2}\right)=\frac{\frac{b}{a}(\cot\beta-\cot\gamma)}{1+\frac{b^2}{a^2}\cot\beta\cot\gamma};$$

or $$a^2\sin\beta\sin\gamma+b^2\cos\beta\cos\gamma+ab\left\{1+\cos(\beta-\gamma)\right\}=0;$$

or $$\frac{\cos\beta\cos\gamma}{a}+\frac{\sin\beta\sin\gamma}{b}+\frac{1}{a+b}=0,$$

and the two like equations in $\gamma$, $\alpha$, and $\alpha$, $\beta$.

Now if the tangents at L, M, N form a triangle whose circumscribed circle is concentric with the ellipse, the radius of this circle will be $a+b$, and we shall have three equations of the form

$$(a+b)^2\cos^2\tfrac{1}{2}(\beta-\gamma)=a^2\cos^2\tfrac{1}{2}(\beta+\gamma)+b^2\sin^2\tfrac{1}{2}(\beta+\gamma),$$

or $$(2b^2+2ab)\cos\beta\cos\gamma+(2a^2+2ab)\sin\beta\sin\gamma+2ab=0,$$

or $$\frac{\cos\beta\cos\gamma}{a}+\frac{\sin\beta\sin\gamma}{b}+\frac{1}{a+b}=0,$$

which are the conditions already found.

Also the equations

$$\cos\alpha+\cos\beta+\cos\gamma=m\cos(\alpha+\beta+\gamma),\quad \sin\alpha+\sin\beta+\sin\gamma=m\sin(\alpha+\beta+\gamma)$$

give for the relation between $\beta$, $\gamma$,

$$2+2\cos(\beta-\gamma)=m^2-2m\cos(\beta+\gamma)+1,$$

or $$2(1+m)\cos\beta\cos\gamma+2(1-m)\sin\beta\sin\gamma+1-m^2=0,$$

which will coincide with the above if $\frac{1+m}{1-m}=\frac{b}{a}$, or $m=\frac{b-a}{b+a}$.

Now, if the tangents at L, M, N form a triangle $lmn$, the coordinates of $l$ are $-(a+b)\cos\alpha$, $-(a+b)\sin\alpha$; hence the coordinates of the centre of gravity of the triangle $lmn$ are

$$-\tfrac{1}{3}(a+b)\cos\alpha+\cos\beta+\cos\gamma,\quad -\tfrac{1}{3}(a+b)\sin\alpha+\sin\beta+\sin\gamma,$$

which are equal to $\frac{1}{3}(a-b)\cos(\alpha+\beta+\gamma)$, $\frac{1}{3}(a-b)\sin(\alpha+\beta+\gamma)$;

hence, the origin being the centre of the circumscribed circle, the centre of the nine points' circle of the triangle $lmn$ is the point

$$\tfrac{1}{2}(a-b)\cos(\alpha+\beta+\gamma),\quad \tfrac{1}{2}(a-b)\sin(\alpha+\beta+\gamma),$$

and the radius of the nine points' circle is $\frac{1}{2}(a+b)$, whence the nine points' circle will touch both the auxiliary circles of the ellipse.

Moreover, denoting $\alpha+\beta+\gamma$ by $\delta$, we have $\delta+\theta=\pi$ (or an odd multiple of $\pi$), whence the equation whose roots are $\alpha$, $\beta$, $\gamma$, is

$$\tan\phi=-\frac{b}{a}\tan\tfrac{1}{2}(\theta-\delta).$$

The equation of the nine points' circle is

$$\left\{x-\tfrac{1}{2}(a-b)\cos(\alpha+\beta+\gamma)\right\}^2+\left\{y-\tfrac{1}{2}(a-b)\sin(\alpha+\beta+\gamma)\right\}^2=\left\{\tfrac{1}{2}(a+b)\right\},$$

and for the points where this meets the ellipse,

$$a^2\cos^2\theta-a(a-b)\cos\theta\cos\delta+b^2\sin^2\theta-b(a-b)\sin\theta\sin\delta=ab,$$

or $\quad a^2\cos\theta(\cos\theta-\cos\delta)+b^2\sin\theta(\sin\theta+\sin\delta)=ab\left\{1-\cos(\theta+\delta)\right\},$

or $\quad -a^2\cos\theta\sin\tfrac{1}{2}(\theta-\delta)+b^2\sin\theta\cos\tfrac{1}{2}(\theta-\delta)$

$$=ab\sin\tfrac{1}{2}(\theta+\delta)\equiv ab\sin\left\{\theta-\tfrac{1}{2}(\theta-\delta)\right\},$$

or $\quad b(b-a)\sin\theta\cos\tfrac{1}{2}(\theta-\delta)=a(a-b)\cos\theta\sin\tfrac{1}{2}(\theta-\delta),$

or $\quad \tan\theta=-\dfrac{a}{b}\tan\tfrac{1}{2}(\theta-\delta).$

Hence, if $\alpha', \beta', \gamma'$ be the three roots of this equation, $\alpha'+\beta'+\gamma'-\delta=0$ (or $2\pi$), *i.e.*, $\alpha'+\beta'+\gamma'=\alpha+\beta+\gamma$. Also the relation between any two of the roots is, as before, found to be (only interchanging $a$ and $b$)

$$\frac{\cos\beta'\cos\gamma'}{b}+\frac{\sin\beta'\sin\gamma'}{a}+\frac{1}{a+b}=0.$$

Now if L′, M′, N′ be these points, and the triangle have its inscribed circle concentric with the ellipse, the usual condition for the inscribed and circumscribed triangle gives the radius of the inscribed circle to be $\dfrac{ab}{a+b}$, and the conditions for this are

$$\frac{1}{a^2}\cos^2\tfrac{1}{2}(\beta'+\gamma')+\frac{1}{b^2}\sin^2\tfrac{1}{2}(\beta'+\gamma')=\frac{(a+b)^2}{a^2b^2}\cos^2\tfrac{1}{2}(\beta'-\gamma'),$$

or $\quad (2a^2+2ab)\cos\beta'\cos\gamma'+(2b^2+2ab)\sin\beta'\sin\gamma'+2ab=0,$

or $\quad \dfrac{\cos\beta'\cos\gamma'}{b}+\dfrac{\sin\beta'\sin\gamma'}{a}+\dfrac{1}{a+b}=0,$

which, coinciding with the relation before formed, proves that the triangle L′M′N′ has its inscribed circle concentric with the ellipse.

Also, as before,

$$\cos\alpha'+\cos\beta'+\cos\gamma'=\frac{a-b}{a+b}\cos(\alpha'+\beta'+\gamma'),$$

$$\sin\alpha'+\sin\beta'+\sin\gamma'=\frac{a-b}{a+b}\sin(\alpha+\beta'+\gamma');$$

whence $\quad \cos\alpha'+\cos\beta'+\cos\gamma'=-(\cos\alpha+\cos\beta+\cos\gamma),$

$$\sin\alpha'+\sin\beta'+\sin\gamma'=-(\sin\alpha+\sin\beta+\sin\gamma),$$

or the distance between the centres of gravity of the triangles L′M′N′, LMN is bisected by the centre of the ellipse.

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**3727.** (Proposed by G. S. Carr.)—Show that the shape of a uniform plane elastic lamina, which will take a circular form when its ends are drawn together to meet, may be obtained by drawing two cutting planes through a tangent which is at right angles to the axis of a cylinder, and unrolling the intercepted part of the surface.

*Solution by* Professor TOWNSEND, M.A., F.R.S.

For, on bending again into its original circular form, and connecting in any manner the two extremities thus brought together, of such a lamina, the conditions for permanence of form, on the usually received principles, are easily seen to be fulfilled; the statical moment of the tangential force developed by its elasticity at the point of junction of the ring, and the moment of inertia of its transverse sectional area, round the axis of flexure, both varying evidently as its breadth at every point; and its curvature, necessarily constant with the constancy of their ratio, being of course the same throughout its entire length.

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**3447.** (Proposed by R. W. GENESE, B.A.)—B, C are points on a circle, A the pole of BC; a series of conics are drawn through ABC touching the circle in a variable point P. Show that all these have the same curvature at the point P, viz., twice that of the given circle.

I. *Solution by the* PROPOSER.

Let TP, TQ be tangents to the circle at P and Q; then it is known that the points A, B, C, T, Q, P lie on a conic; and if Q coincide with P the conic touches the circle, and its circle of curvature at P, being the limit of the circle through T, P, Q, has its diameter equal to the radius of the circle.

II. *Solution by* R. TUCKER, M.A.

Referring the curves to the normal and tangent at P, we have (SALMON, *Conics*, Arts. 239, 241) $ax^2+2hxy+by^2+2gx=0,\ x^2+y^2-2rx=0$ ...(1, 2).

If L be the point where the normal meets the conic again, $PL=-\dfrac{2g}{a}$.

Now let a line through L parallel to the common tangent meet the curve in A′, then by (1), $LA'=\dfrac{4gh}{ab}$; hence the equation to the polar of A′ with respect to (2) is $bx(ar+2g)-4ghy-2bgr=0$ ........................ (3);
and in order that this may be the chord of intersection of (1) and (2), we must have this (by introducing a factor) identical with

$$(a-b)x+2hy+2(g+rb)=0 \quad\text{........................ (4),}$$

by Salmon Art. 239, that is, we must have $2g=-br$ .................. (5);

but (Salmon, Art. 241) the radius of curvature at $P=-\dfrac{g}{b}=\tfrac{1}{2}r$.

Hence the curvature (which varies inversely as the radius of curvature) is equal to twice that of the circle.

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**3345.** (Proposed by Rev. G. H. HOPKINS, M.A.)—Obtain the sum and general term of such series as $3^2+4^2+12^2+84^2+3612^2+$&c. and $11^2+60^2+1860^2+1731660^2+$&c. to $n$ terms; and show that the number of such series, no two of them having any common terms, the first terms

of which are not greater than $a^2$ ($a$ being an odd number), will be $\frac{1}{2}(a-1)-r$, when $r$ is the greatest integer which satisfies $x$ in the inequality $x^2+(x-1)^2<a$.

*Solution by the* PROPOSER.

Every odd number squared is equal to the difference between the squares of two consecutive numbers, the greater of which is the sum of the squares of two consecutive numbers; thus,

$$y^2 = \left\{\tfrac{1}{2}(y+1)^2+\tfrac{1}{2}(y-1)^2\right\}^2 - \left\{\tfrac{1}{2}(y+1)^2+\tfrac{1}{2}(y-1)^2-1\right\}^2 \text{ ...... (1).}$$

Let $y$ be 7 in this expression; then

$$7^2+24^2=25^2,\quad 25^2+312^2=313^2, \text{ ...... } (t_{n-1}+1)^2+t_n^2=(t_n+1)^2;$$

therefore $\quad 7^2+24^2+312^2+48984^2+ \text{......} \; t_{n-1}^2+t_n^2=(t_n+1)^2.$

This series is similar to those the sums of which are required.

The terms are easily found from the relation

$$(t_{n-1}+1)^2+t_n^2=(t_n+1)^2, \quad \text{or} \quad t_n=t_{n-1}(\tfrac{1}{2}t_{n-1}+1).$$

To obtain the general term, since $2t_n+1=(t_{n-1}+1)^2$,

therefore $\quad t_n=\frac{1}{2}\left\{(t_{n-1}+1)^2-1\right\}, \quad$ and $\quad t_n+1=\frac{1}{2}\left\{(t_{n-1}+1)^2+1\right\}$;

therefore
$$t_n = \frac{1}{2}\left\{\frac{1}{2^2}\Big((t_{n-2}+1)^2+1\Big)^2-1\right\}$$
$$= \frac{1}{2}\left\{\frac{1}{2^2}\left[\frac{1}{2^2}\Big((t_{n-3}+1)^2+1\Big)^2+1\right]^2-1\right\},$$

which can be carried on so that the last term may be expressed in terms of the second term of the series.

Since the first term of each series is an odd number, it would seem that the number of series whose first terms are not greater than $a^2$ would be $\frac{1}{2}(a-1)$, no series commencing with unity. This is not the case; for, in the identity (1), if $y=\frac{1}{2}(y'+1)^2+\frac{1}{2}(y'-1)^2$, then

$$y'^2 = y^2 - \left\{\tfrac{1}{2}(y'+1)^2+\tfrac{1}{2}(y'-1)^2-1\right\}^2;$$

or $y^2$ would be the first term of a series, all the following terms of which corresponding with those of another series, the first term of this being less than $y$.

For example, $3^2+4^2+12^2+84^2+\dots$ and $13^2+84^2+3612^2+\dots$ are the same, and $13=3^2+2^2$.

Therefore the number of such series will be reduced by those whose first terms are the sum of the squares of two consecutive numbers squared, which will be given by the greatest value of $x$ in the inequality $x^2+(x-1)^2<a$; or the number will be $\frac{1}{2}(a-1)-r$, where $r$ is the greatest value of $x$ in the inequality $x^2+(x-1)^2<a$.

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**3690.** (Proposed by J. J. WALKER, M.A.)—Prove the determinant identity $(a_1c_2e_3f_4\dots t_r)(b_1d_2e_3f_4\dots t_r)-(a_1d_2e_3f_4\dots t_r)(b_1c_2e_3f_4\dots t_r)$

$$=(a_1b_2e_3f_4\dots t_r)(c_1d_2e_3f_4\dots t_r),$$

the abbreviations of the determinants consisting of the diagonal constituents; all others in the same column being of the same letter, and all of the same line having the same suffix.

I. *Solution by the* REV. W. H. LAVERTY, M.A.

Consider the block of nine rows of seven quantities each

$$\begin{array}{ccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 \end{array}$$

and let (12) denote the determinant formed by omitting the first and second rows, and so on; then we have to prove that the following expression is zero, viz., $(12)(34)+(13)(24)+(14)(23)$.

Out of the left hand side of this equation pick the coefficient of $a_1b_2c_3d_4$; it will be found to be $[12][34]+[13][24]+[14][23]$, where square brackets refer to determinants formed from a block made by omitting the first four rows of the original block. Similar expressions will be deduced for the coefficients of the other products $abcd$; and so the original proposition is reduced to the proof of the same thing for a block of seven rows of five quantities each (the columns having now become rows). In the same way we could reduce it to the proof of the theorem for a block of five rows of three quantities each, and we should then have to prove that

$$(e_5f_6k_7)(g_5h_6k_7)+(e_5g_6k_7)(h_5f_6k_7)+(e_5h_6k_7)(f_5g_6k_7)=0,$$

which is easily established.

Had the original block consisted of even terms, as of eight rows of six quantities each, the final equation to be proved would have been

$$(e_5f_6)(g_5h_6)+(e_5g_6)(h_5f_6)+(e_5h_6)(f_5g_6)=0,$$

which is also easily established.

Precisely similar proofs hold in the general cases.

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II. *Solution by the* PROPOSER.

Express each determinant in terms of its second minors which do not contain any one of the four letters $a$, $b$, $c$, or $d$. Then each side of the above identity will be expressed in terms of the squares and products of identical second minors. Consider the square of one such, suppose $(e_3f_4\dots t_r)$; its coefficient on the left side is

$$(a_1c_2-a_2c_1)(b_1d_2-b_2d_1)-(a_1d_2-a_2d_1)(b_1c_2-b_2c_1),$$

which reduces identically to $(a_1b_2-a_2b_1)(c_1b_2-c_2b_1)$; *i.e.*, to the coefficient of the same squared second minor on the right side. Consider next the product of two second minors, suppose $(e_3f_4\dots t_r)(e_2f_4\dots t_r)$; its coefficient on the left side is

$$(a_1c_2-a_2c_1)(b_3d_1-b_1d_3)-(a_1d_2-a_2d_1)(b_3c_1-b_1c_3)$$
$$+(a_3c_1-a_1c_3)(b_1d_2-b_2d_1)-(a_3d_2-a_1d_3)(b_1c_2-b_2c_1),$$

which reduces identically to

$$(a_1b_2-a_2b_1)(c_3d_1-c_1d_3)+(a_3b_1-a_1b_3)(c_1d_2-c_2d_1);$$

*i.e.*, to the coefficient of the product of the same two second minors on the right side. But it easily appears that the coefficients of any other square or product on both sides would consist of the same terms respectively as in those considered, *with only the same changes of suffix;* so that the identity is completely proved.

**3650.** (Proposed by Professor TOWNSEND, M.A., F.R.S.)—A uniform sphere resting on a rough horizontal plane is set in motion by an impulse applied in a vertical plane passing through its centre; show that, when sliding ceases, the rolling motion will be direct, stationary, or retrograde, according as the direction of the moving impulse intersects its vertical diameter above, at, or below its point of contact with the plane.

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*Solution by* J. J. WALKER, M.A.

Let O be the centre of the sphere, AB the line of action of the impulsive force (P) meeting the radius to the point of contact in B, and let OB $= a + h$, $\angle$ OBA $= \theta$; then, referring to the Proposer's Solution of Professor WOLSTENHOLME's Question 3626, p. 39, $m$ being the mass of the sphere,

$$mV = P \sin\theta - \mu\,(mg + P\cos\theta),$$

and $$\tfrac{2}{5}ma^2\Omega = (a+h)\,P\sin\theta - \mu a\,(mg + P\cos\theta)\,;$$

whence $$ma\,(V - \tfrac{2}{5}a\Omega) = -hP\sin\theta.$$

Now, as shown in the Solution of Question 3626 referred to, when $u = 0$, $v$ is proportional to $V - \frac{2}{5}a\Omega$; consequently the subsequent rolling will be direct, null, or retrograde, according as $h$ is negative, zero, or positive; *i.e.*, according as B is above, coincident with, or below the point of contact of the sphere with the plane.

In what precedes, B is supposed to fall *below* O. If B should be above O, $a$ must be changed into $-a$, and $h$ being essentially positive, $v$ is in this case, too, positive.

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II. *Solution by the* PROPOSER.

Denoting by $v_t$, $w_t$, $u_t$ the linear velocity of the centre of the sphere, its angular velocity round its axis of rotation, and the linear velocity of its point of contact with the plane at any time $t$; by $v_0$, $w_0$, $u_0$, the initial values of the same quantities; by $a$ the radius of the sphere; and by $\mu$ the coefficient of friction; then, since evidently

$$\frac{dv}{dt} = -\mu g, \quad \frac{dw}{dt} = -\frac{5}{2}\frac{\mu g}{a}, \quad v + aw = u;$$

therefore $$v_t = v_0 - \mu g t, \quad w_t = w_0 - \frac{5}{2}\frac{\mu g}{a}t, \quad u_t = v_0 + aw_0 - \tfrac{7}{2}\mu g t\,;$$

and therefore, at the time T when sliding ceases and pure rolling commences, that is at the time T for which $u_T = 0$,

$$w_T = w_0 - \frac{5}{7}\frac{v_0 + aw_0}{a} = \frac{5}{7}\frac{2aw_0 - 5v_0}{a},$$

which will consequently be negative, nothing, or positive, and the ultimate motion of the sphere accordingly direct, stationary, or retrograde, according as $2aw_0 < = > 5v_0$.

Let now F be the magnitude of the impulsive force originating the motion, $b$ the distance (measured downwards) from the centre of the sphere at which its line of direction intersects the original vertical diameter, and $\theta$ the angle that line makes with that diameter; then, since evidently

$$Mv_0 = F(\sin\theta - \mu\cos\theta), \quad M\tfrac{2}{5}aw_0 = F(b\sin\theta - \mu a\cos\theta)\,;$$

it follows at once that $2aw_0 < = > 5v_0$ according as $b < = > a$, and therefore, &c.

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